Labour market frictions and regional disparities

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Abstract

Labour market outcomes differ across regions, and these differences are strongly persistent. The aim of this paper is to explain regional disparities, and the lack of convergence, using labour market frictions. The paper draws on two fields: New Economic Geography, and frictional labour markets. We use the idea of thick labour markets to motivate clustering of economic activity and consequently differences in unemployment rates and wages. Because increasing returns to matching create a link between frictions in the labour market and its size, workers in the thin labour markets will find it increasingly difficult to match with employers in the bigger, more prosperous markets. As a consequence, utility will not be equalised across regions, and regional differences will persist.
JEL codes: J0, J6, R0

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1 Introduction

Labour market outcomes differ across regions. Even within the same country, where language, culture and law should present minimal obstacles, there are stark differences in regional wages and unemployment rates. Perhaps more importantly, these differences persist over time: the regions that suffer from the highest unemployment rates today are very likely to be the same regions that suffered from high unemployment decades ago.

The aim of this paper is to develop a model of regional labour markets that would explain such disparity and its persistence. Instead of assuming various structural differences that could explain why certain regions prosper relative to others, we rely completely on labour markets. We explore whether and how the interaction between firms and workers, and the various frictions that govern these interactions, shape inter-regional differences.

The paper draws on two fields: New Economic Geography, and frictional labour markets. NEG models (started by a series of papers by Fujita, Krugman and Venables in early 1990s) explain cross-regional differences within a single country as the outcome of increasing returns to production. Forces of agglomeration (knowledge spillovers, forward and backward supply linkages) cluster economic activity in one place and are the drivers of regional disparities. In these models, labour markets are usually not modelled explicitly. Migration of workers to the regions of economic activity is assumed to be costless and immediate, and jobs are filled without any delays or frictions. This means that the standard NEG models can teach us only limited amount about the regional difference in labour market outcomes. For instance, in Krugman's (1991) seminal paper, neither region suffers from any unemployment and real wages are equalised across labour markets.

To understand phenomena such as equilibrium unemployment, one has to turn to the the literature on frictions in labour markets (Pissarides (2000)). This model assumes that matching between firms and workers does not happen instantaneously or costlessly. Instead, both sides of the market must spend non-negligible resources and time on finding a match. This gives rise to several real-life phenomena, such as the co-existence of unemployment and unfilled vacancies, and wages that depend on particular match as well as on economy-wide circumstances.

The aim of this paper is to bridge these two strands of literature to model disparities in regional labour markets. In this respect, the paper is similar to the work on local labour markets by Kline and Moretti (2013), Lutgen and der Linden (2015), Beaudry et al. (2014), and Epifani and Gancia (2005).

Our paper differs from this literature in 3 main points.

First, the agglomeration force explored in our paper reflects our focus on labour market. Instead of knowledge spillovers that improve firms' productivity, or supply linkages that make it less costly to produce, we focus on the idea of thick labour markets: firms in a larger labour market are more productive because the matching function is more efficient, which allows them to spend less time and resources on recruiting workers.

Second, we assume that migrating workers face significant costs. This, coupled with labour market frictions, means that workers will not always find it optimal to move between regions. This will create persistent regional differences in wages and unemployment rates, and slow down agglomeration.

This is not the first model to predict that regions might reach an intermediate stage of divergence (i.e. that not all economic activity must move to single location). New Economic Geography models predict that economic activity will cluster until the marginal increase in productivity is outweighed by the greater transport costs. Similarly, the literature on local labour markets assumes that the declining value of amenities per inhabitant and the increasing housing costs will put a natural stop to regional divergence and agglomeration. In these models, the process of regional divergence will stop at an optimal level, and utility across regions will be equalised.

This is not the case in our model. We do not include any transportation costs or amenities. Instead, the obstacle to agglomeration is the same as its source: increasing returns to matching. Depending on the parameters of the model, frictions in the labour market may make it unprofitable for firms to hire from the smaller regions despite the lower local wages. The process of divergence will be halted, but the outcome will not be optimal, and utilities will not be equalised across regions. In our model, it does matter which region the worker resides in.

The next section presents some stylised facts that motivate this paper. Section 3 reviews the literature. In section 4, we present a simple Pissarides-style model of labour markets with increasing returns to matching and exogenous population. In section 5, we turn to workers' and firms' location decisions, and then use these to identify steady states when population is endogenous (section 6). As a part of our analysis, we run some numerical simulations of our model. Section 7 concludes.

2 Stylised facts

In this section, we document the extent and features of disparities in local labour markets. We focus on a sample of six major developed countries: USA, UK, Germany, Italy, France and Spain, in the years 1999-2014. We use OECD regional statistics to pin down trends in unemployment, incomes, and migration across regions. Specifically, we look at Territorial Level 2 (by OECD classification) regions, which correspond to

- 22 *régions* in France
- 16 *lander* in Germany
- 21 *regioni* in Italy
- 17 *comunidades autonomas* in Spain, excluding the cities of Ceuta and Melilla on African coast
- 12 *regions and countries* in the UK
- 51 *states* in the US, including the District of Columbia

The stylised facts presented below are in line with the empirical work on regions disparities. Moretti (2011) and Kline and Moretti (2013), and Overman and Puga (2002) document the extent and persistence of disparities across US metropolitan areas and transnational EU regions, respectively. Moretti (2011) gives evidence on nominal and real wages, productivity and innovation, while Overman and Puga (2002) highlight the persistence of regional differences in wages and unemployment. Kline and Moretti (2013) highlight that a substantial regional dispersion in wages and unemployment rates persists even after controlling for local human capital. They also make the point of comparing variation in unemployment rate over the business cycle with differences across regions at a given point in time. Their conclusion, replicated in this paper in table 1, is that the former is often smaller than the latter. Several papers suggest that these regional differences are growing over time (Moretti (2011), Overman and Puga (2002)).

Blanchflower and Oswald (1995) introduced the wage curve, an "empirical law" of positive co-movement of wages and unemployment rates across regions. Because workers are documented to move into regions with higher wages and lower unemployment (Beaudry et al. (2014)), regions with greater agglomeration are going to experience positive net inflows of labour. This has also been shown empirically (e.g. Pissarides and Wadsworth (1989) for the UK).

2.1 Existence of regional differences in labour markets

The differences in income and unemployment rates across regions are well documented. There is the North-South divide in the UK, Spain and Italy; the East-West division in Germany, the Rust Belt states in USA; and clustering of unemployment and low GDP per capita in the northern and southern coastal areas in France.

In our sample, the difference between the highest and lowest unemployment rates across regions within a single year can easily reach double digits. The difference was more than 20 percentage points in early 2000s in Italy, and as much as 10 percentage points in the US with what is traditionally believed to be a much more flexible labour market.

These differences are often greater than the variation in the national unemployment rate over the business cycle. In our sample, in one case only was the average spatial variation smaller than the variation over the business cycle. Even in countries such as the UK and the US, regional differences were at least as big as differences over time. The special case, as can be seen in table 1, is Spain that underwent a very severe recession in 2008-2013.

The large large variation in unemployment rates across regions persists even after controlling for the underlying differences in human capital of the populations. In the left hand sides panel of figures 1 and 2, we plot distribution of regional unemployment rates for the 6 countries in year 2014 (2013 for the US) in light blue, and the same distribution but controlling for education levels in red (we superimposed the mean of the original distribution on the conditional distribution for comparison). Controlling for education indeed changes the distribution, and removes some of the variation, but the differences nevertheless persist.

Regional variation in per capita income (our proxy for regional wages) follows similar patterns. Table 2 documents the extent of income disparity over time and across regions in the 6 developed countries; as with unemployment rate, the difference in income over the business cycle is comparable or smaller to the variation in disposable income across regions. In figures 3 and 4, we control for variation that arises from human capital differences, but the large disparities prevail.

Table 1: Unemployment rates across regions and time, 1999-2014

country	**national average**	**highest vs. lowest, across regions**			**highest vs. lowest, over time**
		min	**max**	**average over time**	
France	9.2	6.0	18.5	8.9	4.2
Germany	8.0	6.9	16.2	12.0	6.2
Italy	9.1	10.3	25.9	16.9	6.6
Spain	14.2	7.9	19.7	14.8	17.9
UK	6.2	3.0	6.3	4.1	3.4
USA	5.9	3.6	10.6	5.9	5.6

2.2 Persistence of regional differences

The large differences in incomes and unemployment rates between regions persist over time. As figure 5 shows, the regions with high unemployment in the past are likely to suffer from high unemployment in the present: regional unemployment rates in 1999 can explain between 42% (France) and 97% (Italy) of regional unemployment rates in 2014.

This persistence is not just absolute (high-unemployment regions are more likely to suffer from high unemployment in the future), but relative, too. Figure 6 plots regional unemployment rate over time for Italy, Germany and the UK (the other three countries have too many regions

Table 2: Differences between lowest an highest annual disposable per capita income levels across regions and time, local currency, 1995-2011

country	**national average**	**highest vs. lowest, across regions**			**highest vs. lowest, over time**
		min	**max**	**average over time**	
France	17267.46	5642	6797	6385.1	6016.0
Germany	17662.9	4333	6037	5085.6	5227.0
Italy	15680.2	7839	9804	9093.8	4095.0
Spain	11848.9	4302	8186	6508.9	6132.0
UK	14544.7	5061	8177	6541.2	6607.0
USA	33243.6	14518	16686	26819.9	16024.0

to plot comfortably). We can see that the regions that suffer from the highest unemployment rates at the start of the period (e.g. North East England in the UK, or Saxony-Anhalt in Germany) maintain their position in the unemployment-rate ranking 15 years later. Similarly, the three regions with the lowest unemployment rates are still doing best relative to the rest of the country in 2014 (such as the Italian provinces of Bolzano-Bozen and Trento). The worst-off regions do react to aggregate shocks, but even if their unemployment rate is falling, they never fully converge to the best-off regions. Germany, with its impressive long-term reduction in unemployment across the country, is a good example of this.

This persistence extends to incomes. Figure 7 depicts correlations between average annual disposable income in a region between the years 1995 (for most countries) and 2011. For all six countries in our sample, the predictive power of 1995 income is at least 82%.

2.3 The wage curve

The wage curve is an empirical "law" documented by Blanchflower and Oswald (1995) which states that in a cross-section of regions, wages are negatively correlated with unemployment rates. This is also a feature of our sample. In figures 11 and 9, we correlate average annual disposable income with unemployment rate across regions. We can see that the regions with higher unemployment rate also suffer from lower incomes. As the right hand side panel shows, this relationship holds even after controlling for differences in regional education attainment. Barring any amenities that would make living in the high-unemployment, low-income regions more attractive, the wage curve relationship suggests that utility is greater in some regions than in others.

2.4 Inter-regional migration

Inter-regional migration is an important potential channel of equilibrating regional differences. In the absence of any legal barriers to movement, and given that cultural and language barriers are likely to be much smaller than in the case of international migration, we would expect movement of people from the less to the more prosperous regions of the country.

Our data shows that the proportion of population that moves within country every year is on average less than 2% - and not all of this movement is likely to be die to labour market reasons (table 3).

Looking at migration patterns, we can see from figure 10 that people indeed tend to move into regions with higher employment rates (our measure of job-finding probability) and higher wages.

2.5 The role of population density

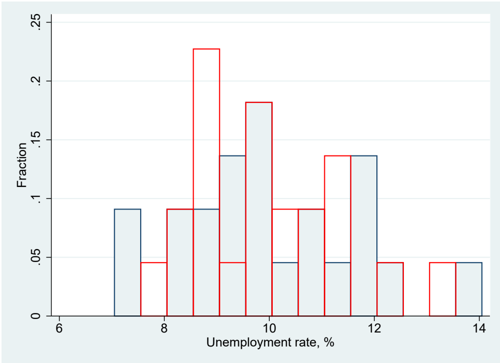
The combination of the wage curve relationship, trends in regional differences, and the patterns of inter-regional migration suggest that population is clustering in regions with higher incomes and lower unemployment. To test this hypothesis, we included a measure of workforce density (population of working age per unit of area) in wage curve regressions. Including this variable also relates to the wealth of literature exploring agglomeration effects, where places that have large population are also the ones with higher productivity and hence higher wages and lower unemployed rates (Roback (1982)).

Table 11 summarizes the results for the 6 countries in our sample. We can see that workforce density is indeed positively associated with greater income, and in France, the UK and USA is this effect significant. Of course, this is a correlation. This paper suggests one possible explanation (and direction of causation) of this relationship.

The persistence of the patterns of regional differences suggest that that this clustering does *not* lead to equilibration between regions. This could be also seen in figure 10, where net migration inflow today positively depends on employment rates and average incomes in the previous period.

Table 3: Average annual inter-regional migration

country	**inter-regional migration as a share of total population**
Italy	0.6%
Spain	0.8%
Germany	1.3%
USA	1.1%



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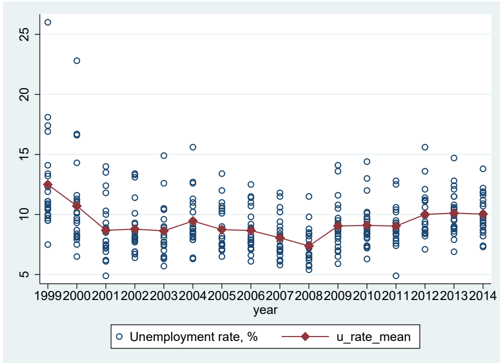

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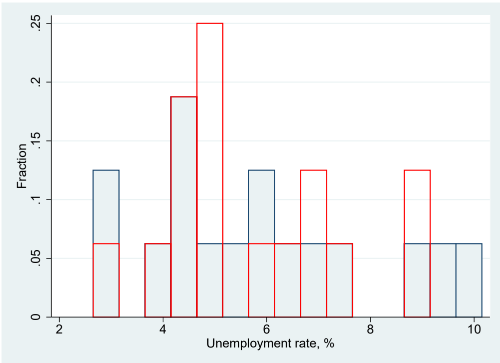

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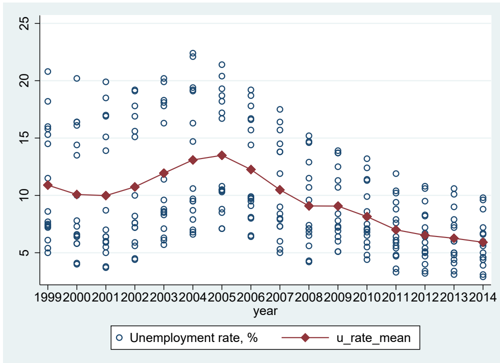

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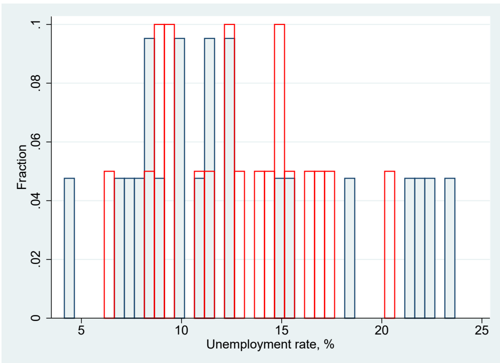

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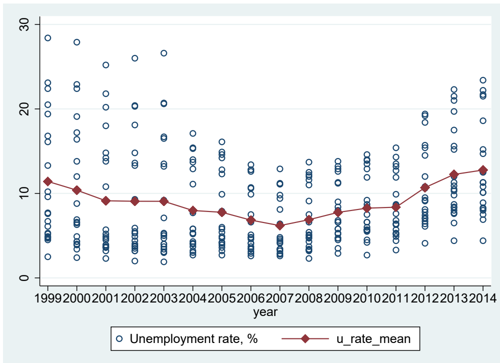
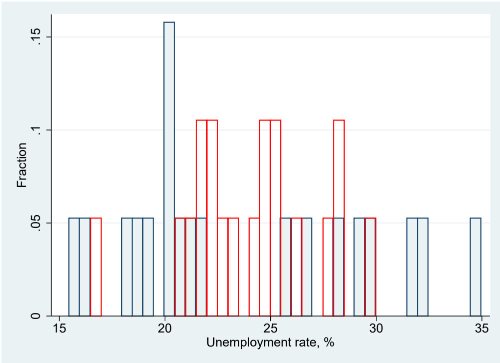


(f) Italy

Figure 1: Unemployment rates across regions and over time. Left hand side: blue bars represents unemployment rates, red bars unemployment rates conditional on education.



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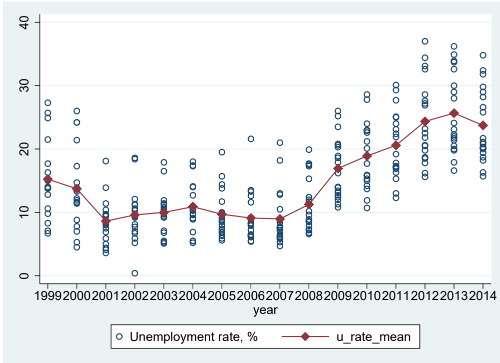

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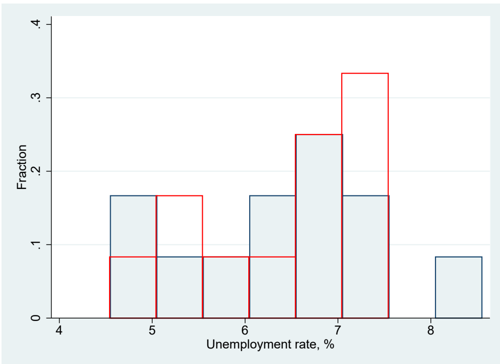

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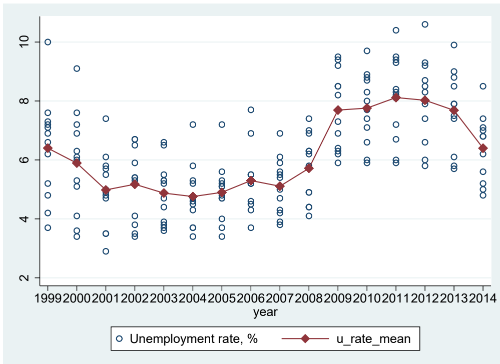

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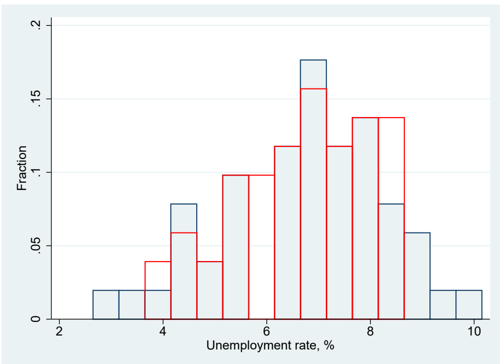

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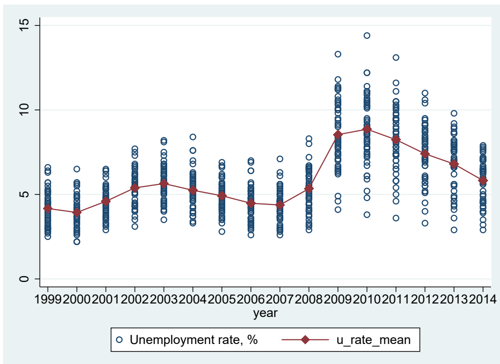

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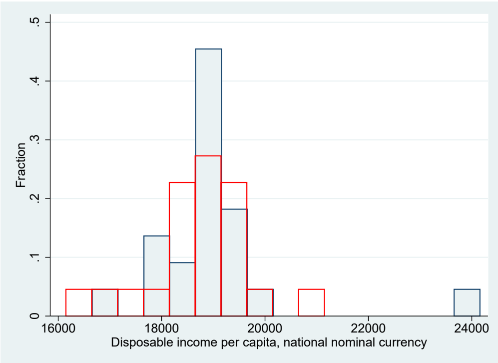
Figure 2: Unemployment rates across regions and over time. Left hand side: blue bars represents unemployment rates, red bars unemployment rates conditional on education.



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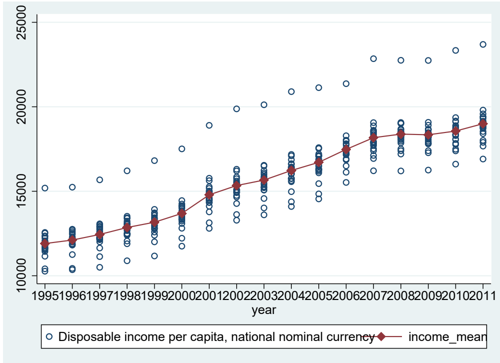

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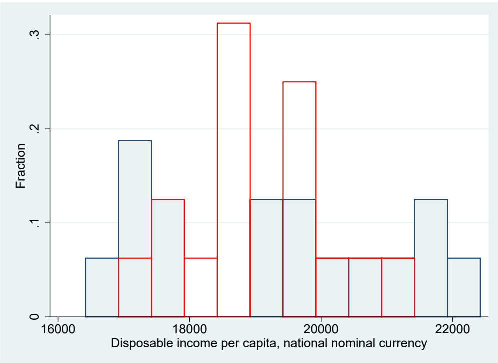

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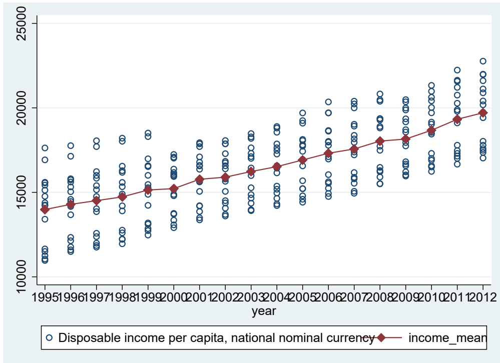

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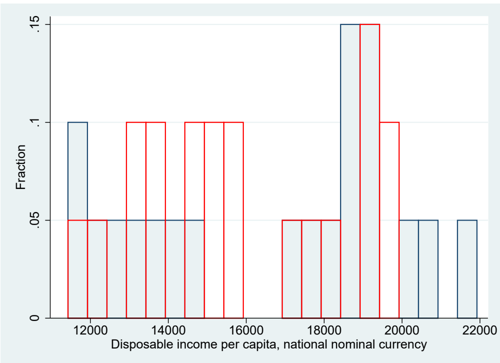

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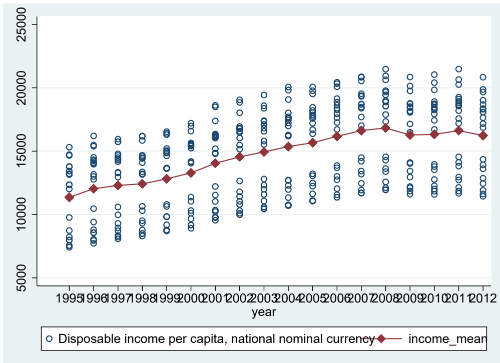

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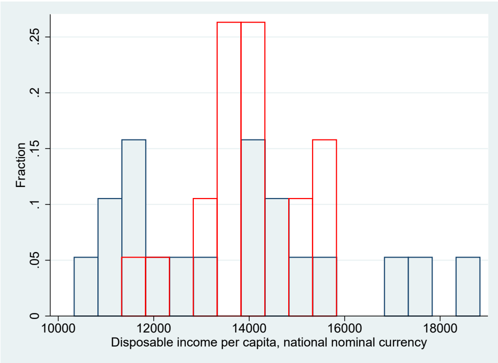
Figure 3: Disposable per capita income across regions, year 2011. Left hand side: blue bars represents unemployment rates, red bars unemployment rates conditional on education.



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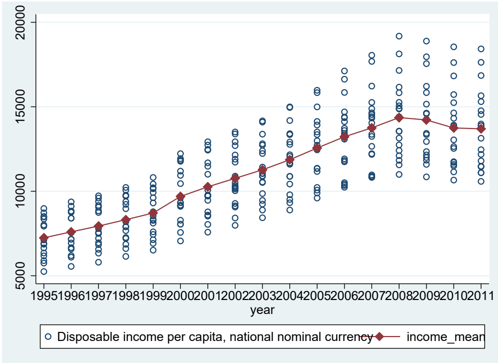

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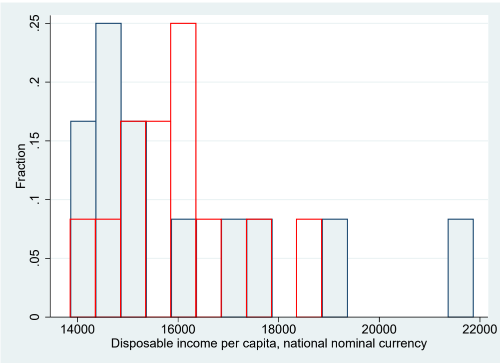

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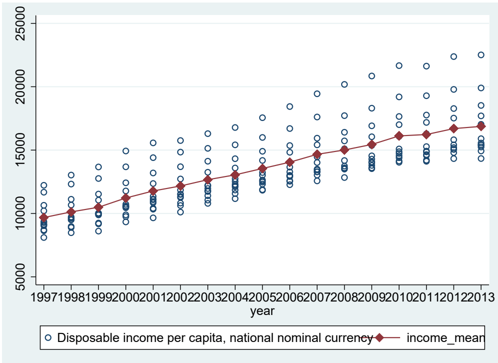

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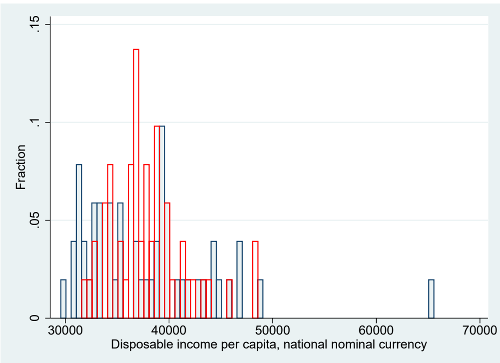

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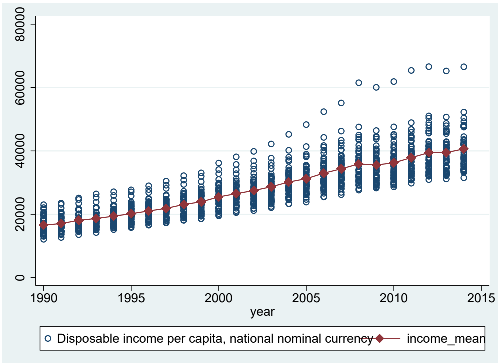

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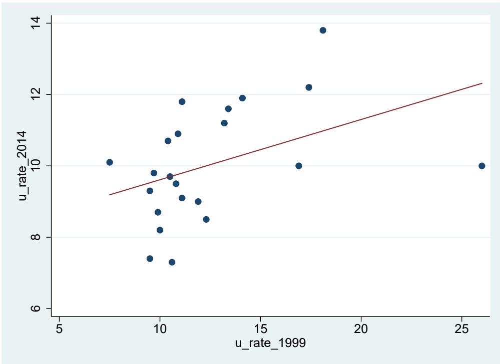
Figure 4: Disposable per capita income across regions, year 2011. Left hand side: blue bars represents unemployment rates, red bars unemployment rates conditional on education.



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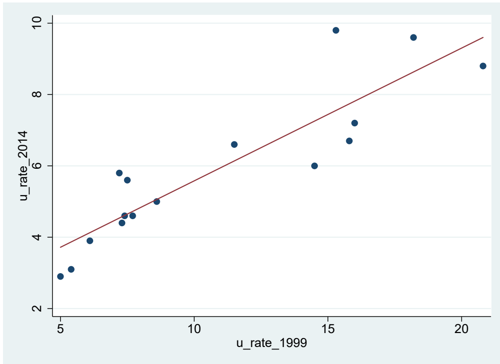

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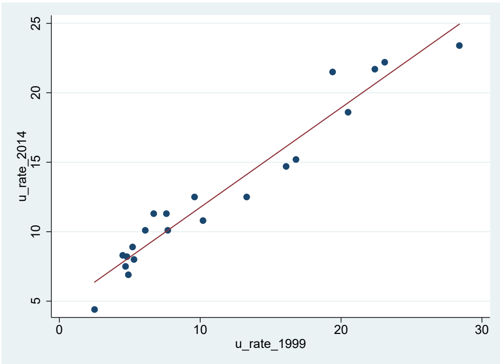

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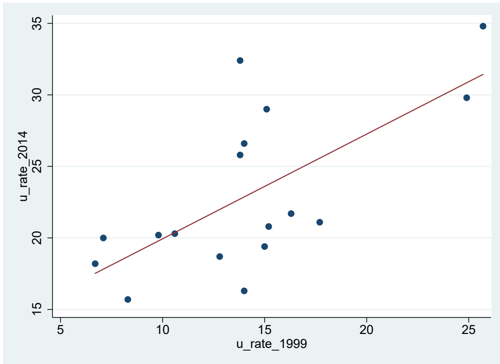

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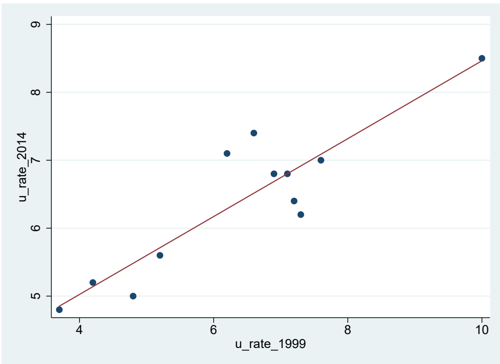

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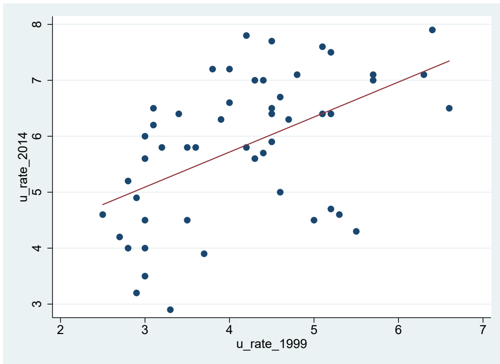

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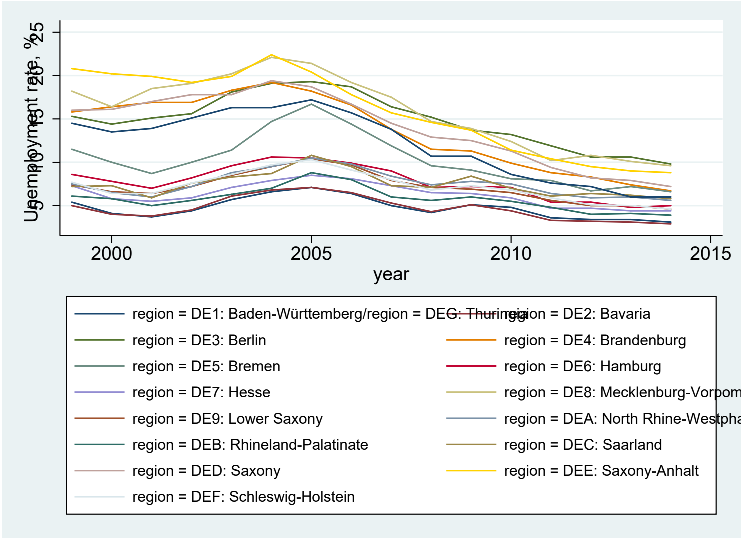
Figure 5: Correlation between regional unemployment rate in 1999 and 2014.



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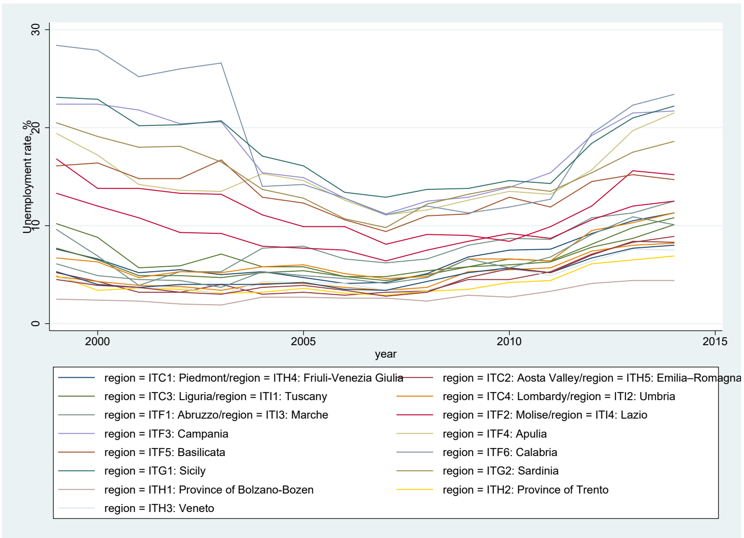

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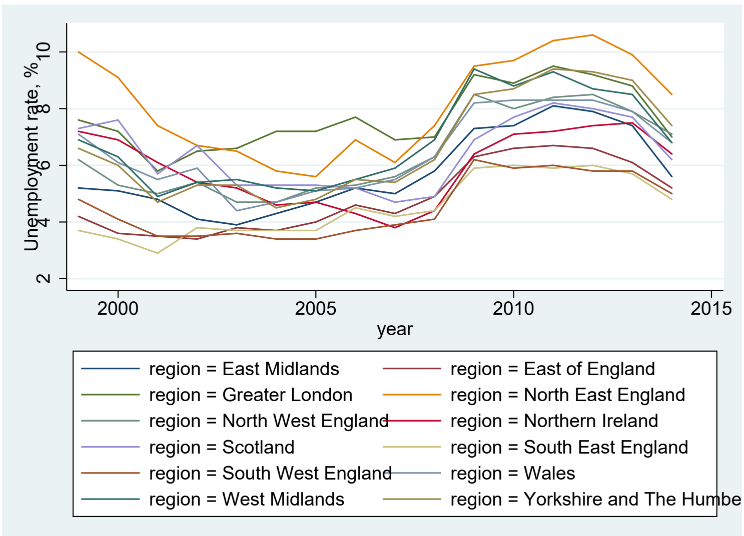

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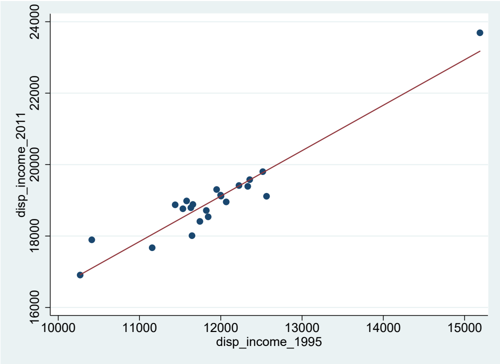
Figure 6: Regional unemployment rates over time.



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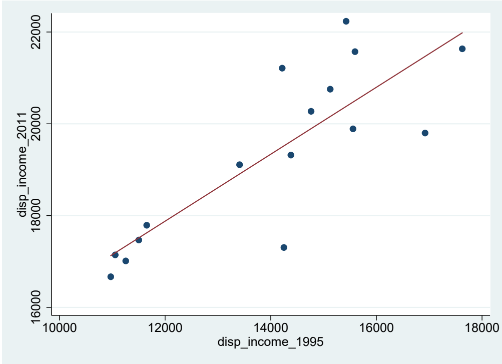

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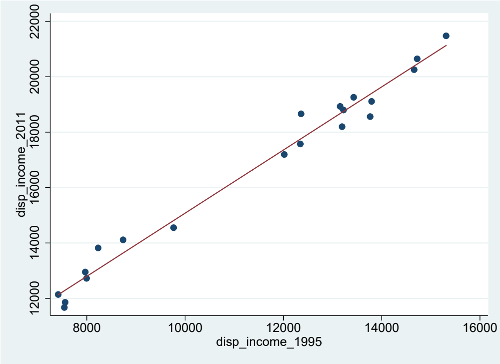

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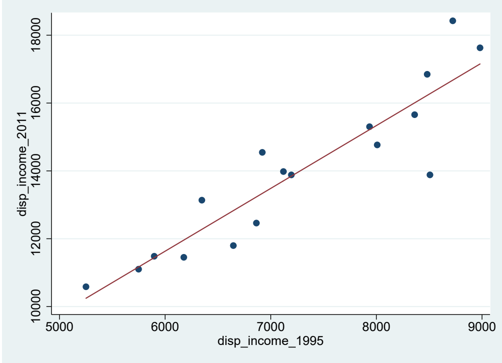

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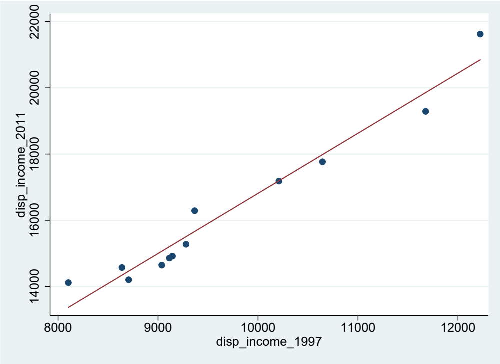

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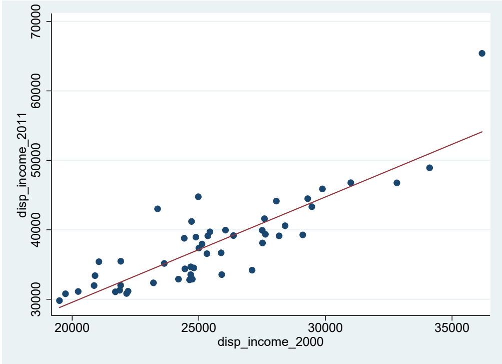
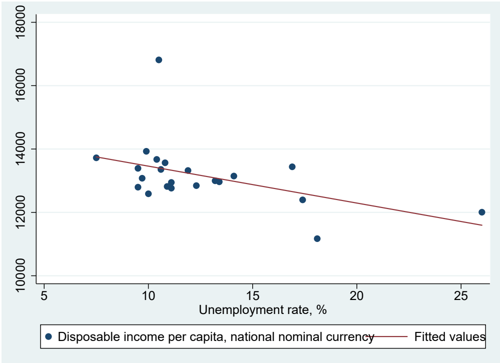
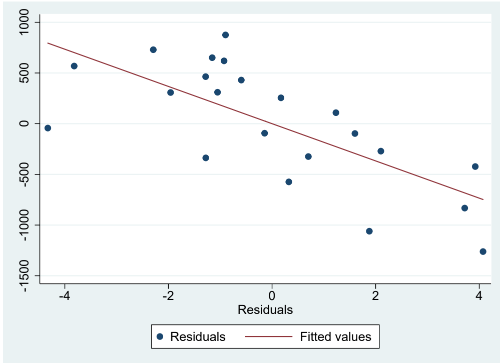


(f) USA

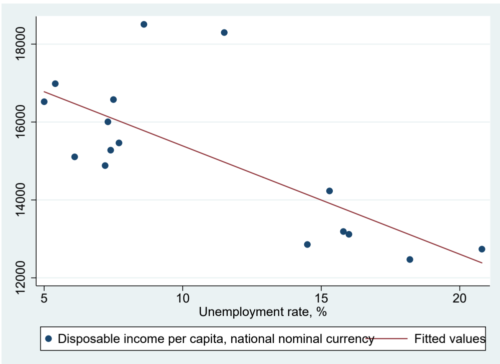
Figure 7: Correlation between regional disposable incomes in 1990s and 2011.



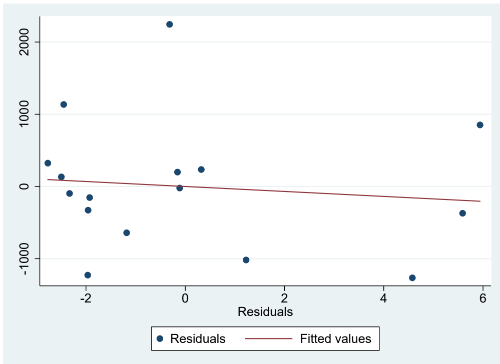
(a) France



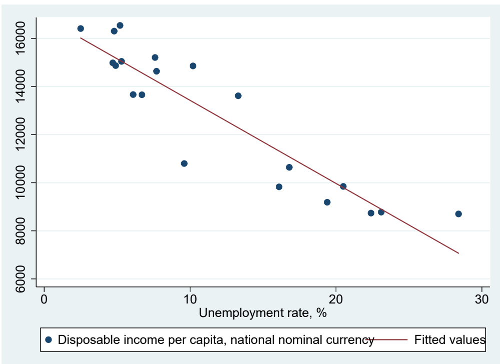
(b) France



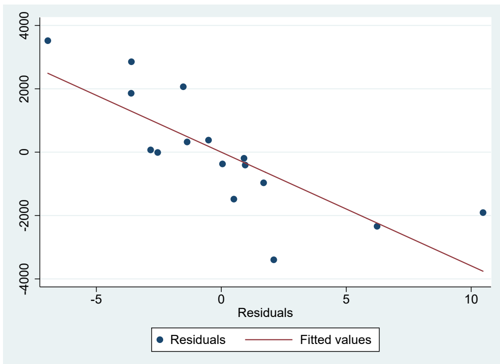
(c) Germany



(d) Germany

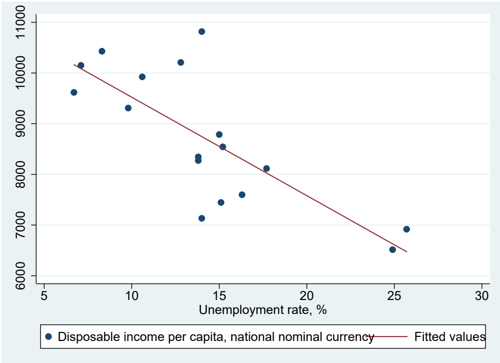


(e) Italy

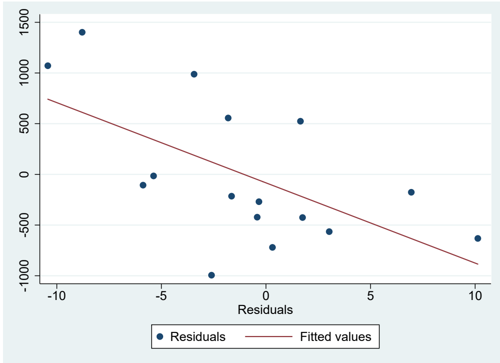


(f) Italy

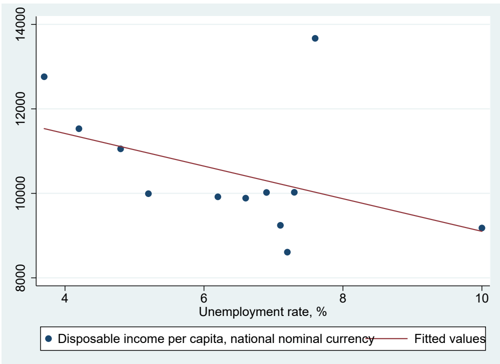
Figure 8: Unemployment rate (x-axis) against disposable income (y-axis). Right hand side is after partialling out the effects of education.



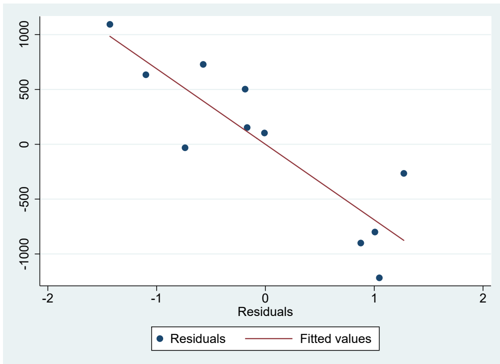
(a) Spain



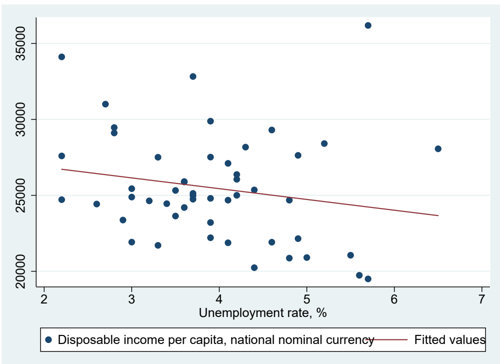
(b) Spain



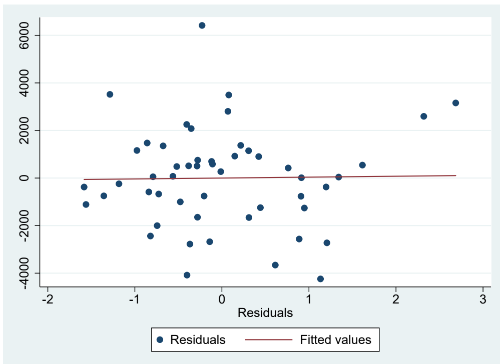
(c) UK



(d) UK

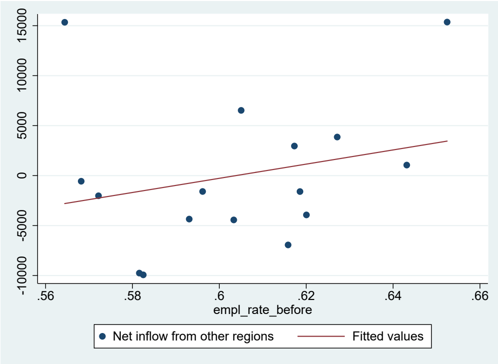


(e) USA

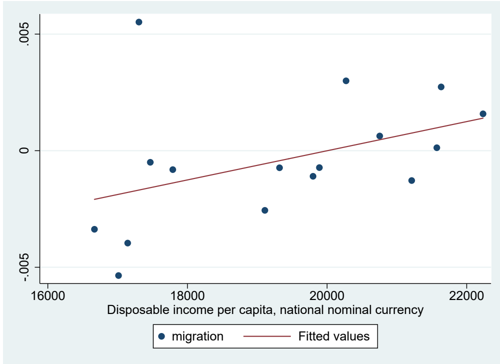


(f) USA

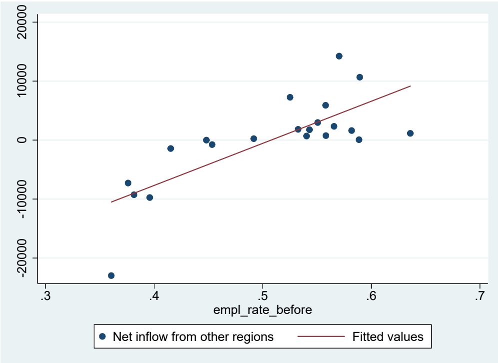
Figure 9: Unemployment rate (x-axis) against disposable income (y-axis). Right hand side is after partialling out the effects of education.



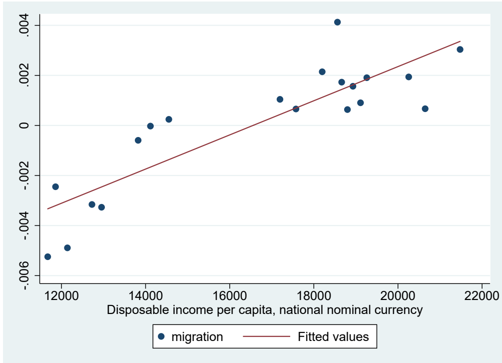
(a) Germany



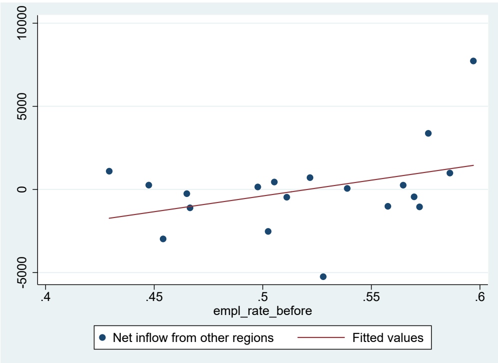
(b) Germany



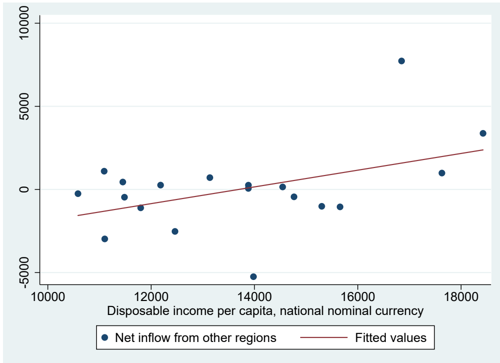
(c) Italy



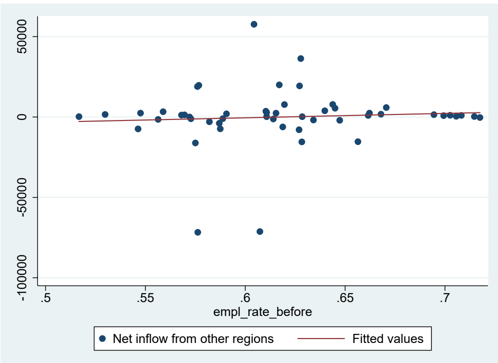
(d) Italy



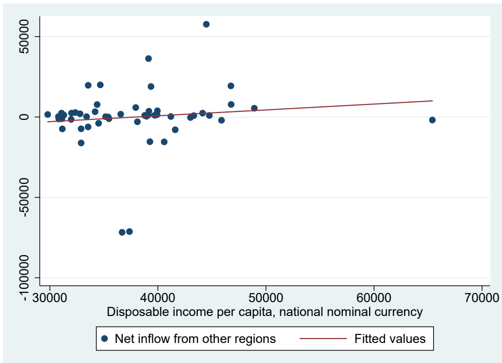
(e) Spain



(f) Spain



(g) USA



(h) USA

Figure 10: Migration patterns in 2011. Left hand side: correlation between net inflow from other regions and employment rate. Right hand side: correlation between net inflow from other regions and average annual disposable income.

	Italy	Spain	France	Germany	UK	USA
			log of disposable income			
log(unempl_rate)	-.0796673***	-.10081809***	-.03014399***	-.05870427***	-.11362134***	-.1384205***
primary_educ	-.00195645*	-0.00299367	-0.00149498	0.00284754	.00290183*	-0.00101095
secondary_educ	0.00129734	-0.00005376	-0.000056	-0.00095967	0.00086283	0.00032514
tertiary_educ	-0.00058336	0.00534028	0.00001092	-.00404588*	0.00164494	.00802191*
N	261	219	283	222	173	714
R-squared	0.6213	0.8323	0.7868	0.751	0.5721	0.7139

	Italy	Spain	France	Germany	UK	USA
			log of disposable income			
log(unempl_rate)	-.07906383***	-.09967341***	-.02878779***	-.0559557***	-.11611387***	-.14269701***
primary_educ	-.00191924*	-0.0022951	-.00190261*	0.00203327	0.00219528	-0.00146804
secondary_educ	0.00127326	0.00003925	-0.00039011	-0.00096164	0.00059616	0.00011822
tertiary_educ	-0.00055886	.00549036*	-0.0009653	-.00484775*	0.00054724	.00443338*
workforce_density	0.04948046	0.05811469	.30322962**	0.02269315	.06203284***	.18507978***
N	261	219	283	222	173	714
R-squared	0.6181	0.8307	0.8883	0.7628	0.7915	0.7532

Figure 11: Wage curve regressions. Dependent variable: log of average per capita annual disposable income. Explanatory variables: log of unemployment rate, workforce per unit of area, share of population with primary/secondary/tertiary education as their highest qualification. Random-effects regressions with unreported constant and year dummies. Years: 1995-2011.

3 Literature review

The aim of this paper is to contribute to our understanding of variation in wages and unemployment across regions within the same country. We follow the growing research on local labour markets, which is a synthesis of two strands strands of literature: New Economic Geography (NEG), and frictional labour markets.

The ideas of spatial agglomeration were first discussed in the seminal work of Fujita, Krugman and Venables (Fujita et al. (2001); Krugman (1991); Krugman and Venables (1996)). These papers model geographic clustering of economic activity as the interaction between three factors: transportation costs, increasing returns to production, and mobility of inputs. Firms that locate in the same city/region can take advantage of the same supply chains, which makes them more productive. Hence, when transportation costs are positive but not so sizeable as to prevent firms from locating away from their customers, it will be profitable for companies to reap these economies of scale by locating where other firms are located. This is what drives regional economic differences. Due to the presence of increasing returns to scale, the system exhibits multiple equilibria and strong path dependence. There is nothing that differentiates a peripheral region from the economic core of a country but the cumulative impact of agglomeration forces.

An important assumption in NEG literature is one of perfect labour mobility. This is justified in several ways: NEG models usually look at regional differences within a single country, where barriers to movement of people are minimal. Even if there are some delays and obstacles in inter-regional migration, these are thought to be insubstantial in the long-term view taken by these models. As a result, workers always follow the firms, and there is no unemployment (regional or otherwise).

Search and matching models, on the other hand, focus exclusively on capturing the dynamic interactions within a labour market. Even in the presence of unemployed workers willing to work, and attractive job vacancies, a match between the two sides of the market are not taken for granted. Following the canonical Diamond-Mortensen-Pissarides model (summarised in Pissarides (2000)), the frictions in the labour market that prevent workers and firms to meet immediately and costlessly are modelled by a matching function that at any point in time matches only a fraction of those searching in the labour market. Intuitively, observable and (especially) unobservable worker and firm heterogeneity, asymmetric and incomplete information, and things such as moving to a different location all explain why it takes time and resources for employers and workers to meet and establish contracts. As a consequence, the labour market never clears in the classical sense of the word, and the equilibrium is one where unfilled vacancies and unemployed workers coexist alongside each other.

There is a growing body of literature that tries to marry these two approaches to explain regional variation in labour market outcomes such as unemployment rate, labour market participation, and wages. The seminal contribution to the literature on local labour markets was Roback (1982). It demonstrated how to set up and solve a tractable model of a regional economy where rents and wages are determined simultaneously. Instead of assuming economies of scale as the driving factor of regional differences, regional variation was explained by differences in local amenities that entered both the production function and workers' utility. A Labor Economics Handbook chapter by Moretti (2011) presents an expanded version of this seminal model. Moretti allows workers to have preferences over locations, which means that labour may not be perfectly flexible; and considers the case of adjustable housing supply, so that housing supply is not perfectly rigid. He also expands the model to include worker heterogeneity by skill, and the existence of non-tradeables along with tradeable goods. All of this allows for a more realistic analysis of labour market equilibrium and the welfare incidence of labour market shocks, which is useful for policy considerations. Moreover, Moretti motivates the assumed exogenous productivity differences across cities by agglomeration forces, linking agglomeration to local labour markets.

However, models a la Roback and Moretti tells us nothing about regional unemployment. In both models, the focus is on wages: workers always provide one unit of labour and the labour market always perfectly clears. Yet, a robust positive relationship between regional unemployment, wages and productivity has been documented in many studies (Blanchflower and Oswald (1995) described the positive correlation between local wages and unemployment rate "an empirical law").

One of the first papers to model and explain regional unemployment was Epifani and Gancia (2005). They combine NEG models of spatial agglomeration with search and matching in labour markets. Just like in the standard NEG models, the firms residing in the agglomerated region enjoy higher productivity, which results in higher demand for labour. When worker mobility is perfect, they will move to take advantage of higher wages until the point at which the marginal congestion externalities makes them indifferent between the economic core and periphery. Epifani and Gancia refine this model by specifying firm's job creation and allowing for unemployment to exist in equilibrium. The periphery region, where forces of agglomeration are weak, experiences less vacancy creation which leads to higher unemployment. It is thus the same force that causes clustering of economic activity that also explains regional unemployment differences. The presence of significant congestion externalities (here modelled as amenities per person) means that workers' utility in steady state is eqaualised across regions, despite the potentially significant disparities in wages and unemployment. It also means that the process of regional divergence will optimally settle in a state of partial divergence, halfway through between no regional differences and all economic activity taking place in one region only.

There are several papers that follow in similar vein. Francis (2009) extends this model by endogenising job destruction. He shows that even though job destruction is higher as a result of the creative destruction process in more agglomerated regions, this negative effect on local employment rate is more than mitigated by the positive impact of greater job creation. Zierahn (2013) combines NEG model of regional production with a different model of frictional labour markets, that of efficiency wages, to analyse the impact of frictions in wage-setting on regional unemployment. Similarly to Epifani and Gancia (2005), unemployment rates are lower and wages are higher in the agglomerated regions: in Zierahn's model, the higher productivity that results from agglomeration allows firms in the economic core to pay more to its workers and thus elicit greater work effort. In the periphery, on the other hand, wage-setting frictions are greater, which causes unemployment to be higher.

Kline and Moretti (2013) arrive at similar conclusions but using the simpler framework of local labour markets. While NEG models usually have to be solved numerically, building on a standard local labour market model a la Moretti (2011) means that closed-form analytical solutions are available. Just like in the original model, there is a spot market that determines the supply of housing, and cities (or regions) differ in their amenities and productivity. Because workers are free to move, utility will be equalised across geographical units. However, the existence of exogenous productivity differences means that labour demand varies over cities, which gives rise to wage and unemployment differences: more productive cities have higher wages, lower unemployment, and higher rents. Higher migration into more productive cities also means that their population will be greater. While this model remains silent on the causes of varying city productivity, it is a useful tool for thinking about place-based policies in relation to labour market frictions.

The papers cited so far all model local labour markets as separate entities connected only by trading and migration of workers: for a worker and an employer to meet, they must reside in the same area. Beaudry et al. (2014) relax this assumption. They set up a standard model of local labour market, with exogeous productvitiy differences across cities and search and matching frictions in the labour market, but allow workers to receive job offers from their home city as well as from other cities. A worker can emigrate only if she accepts such a job offer. Even though the authors assume that these migration shocks are sufficiently frequent to equalise utility across locations, their setup is the first (to our knowledge) that links the efficiency of the matching mechanism to the strength of migration flows.

Lutgen and der Linden (2015) also allow regional labour markets to interact directly. They extend Moretti (2011) to include the possibility of worker search over multiple regions (i.e. using internet). This paper focuses less on explaining variance in regional labour market outcomes and more on efficiency of multi-region search. They find that because multiple-region searches introduces new congestion externalities in the labour market, the resulting efficiency contribution of internet searches is mostly negative. In the absence of any regional differences, regional variation in unemployment is fundamentally driven by workers' preferences over location.

A recurring theme in the existing literature is that a worker's utility is independent of her place of residence. Higher wages and lower unemployment rates are offset by higher housing costs and negative congestion externalities. As long as we assume that workers can move freely between regions, they stay in a periphery region only when they are compensated by better amenities or lower costs of housing. Boadway et al. (2004) is one of the few papers on local labour markets that assumes costs to migration. In particular, the assumption is that labour is perfectly immobile, and the spatial action instead moves to firms' location and job creation decisions. The search and matching frictions, together with different incentives for firms in terms of job creation, generate regional variation in unemployment. However, the authors also assume perfect unemployment insurance, so that differences in wages and unemployment rates only matter for efficiency, not equity.

The aim of our paper is to present an alternative way to model economic clustering and regional variation in wages and unemployment. The biggest difference is the source of agglomeration forces. Current literature takes one of three paths: it either assumes exogenous productivity differences across cities/regions (e.g. Moretti (2011)), models these as the result of transportation costs and forward and backward linkages (Epifani and Gancia (2005), Francis (2009), Lutgen and der Linden (2015)), or considers knowledge spillovers. We focus on increasing returns to labour market matching. This is by no means a new idea: the possibility of thick labour market or labour market pooling effect originate in Marshall (Krugman (1991)), and have been explored in other parts of labour economics (Ellison et al. (2014) analyse the behaviour of a standard search and matching model as returns to scale change from decreasing to constant to increasing). Yet, so far has been largely avoided when it comes to using it to explain regional disparities (although there has been some work done using increasing returns to scale in matching in urbanisation literature, see for example Gautier and Teulings (2009)).

The choice of agglomeration force reflects the paper's focus on labour market mechanisms. In this respect the work is similar to Kline and Moretti (2013); but unlike most of the literature, we abstract from any amenities or housing markets. The main reason for doing so is that it allows us to focus solely on labour market frictions and their role in regional divergence. In our paper, partial divergence is an equilibrium despite the absence of compensating differentials that would motivate workers in the periphery to stay there. Instead, it is the size of labour market frictions that prevent them from moving. Closely linked to this is our interpretation of persistent regional differences as the result of malfunctioning markets rather than an optimal strategy.

Our inclusion of migration costs plays a big role in this narrative. In a contrast to increasing returns in matching, imperfect mobility is a fairly standard assumption in the literature: the seminal work by Moretti (2011) included an idiosyncratic location preference term in its utility specification, which resulted in some workers staying in the economic periphery despite the higher wages in the core, and this assumption is present in much of the subsequent research (Beaudry et al. (2014) work with preferences over city-specific amenities). In this paper, migration costs are the same for all workers. The main difference, however, is in the interpretation: location preferecens mean that workers could not be made better off by re-locating to a different region, while the imposition of migration costs means that workers would like to move but cannot. In this respect, our paper is similar to Boadway et al. (2004) that assumed infinite migration costs.

As a consequence, utility is not equalised across space even in a steady-state equilibrium. This is in a stark contrast to most literature where spatial equilibrium is defined precisely as one where workers' migration has exhausted all possible improvement in utility. The empirical evidence on this assertion is ambiguous. Beaudry et al. (2014) estimate elasticity of migration with respect to regional differences in wages and employment, and show that workers respond up to three times more strongly to unemployment than to wages; in fact, the impact of wages on migration is statistically insignificant. There is little sign of age convergence within industries, and they fail to find persuasive evidence that house prices adjust to reflect net migration from US cities, which further undermines the assumption that utility of the marginal worker is equalised across space.

Amenities are equally elusive in empirical data. While housing prices are more straightforward to capture, the actual relationship between congestion and amenities may go either way. In her seminal paper, Roback (1982) actually estimates amenities to increase with population density.

Arguably, empirical support for increasing returns to matching is similarly inconclusive. Pissarides et al. (1986) and Blanchard and Diamond (1989) ran regressions of number of matches on the number of vacancies and unemployed for the aggregate economy; both were unable to reject the null hypothesis of constant returns to scale. On the other hand, similar research by Yashiv (2000) and Warren (1996) find evidence of economies of scale. Petrongolo and Pissarides (2006) point out that hiring rate depends not only on the rate of meetings between firms and workers, but also on wages. In particular, the upward wage adjustment may completely offset the impact of higher meeting rate, so that increasing returns would not be detected from hiring data only. The authors thus separately estimate a wage equation and a workers' hazard rate. They find economies of scale effect in wages, but not in the matching function. Baker et al. (1996) focus on the potential mis-measurement of the number of searching workers which arises when studies ignore on-the-job search. Using a Canadian dataset, they demonstrate that correcting for this under-measurement reveals significant economies of scale in the matching function. Their paper is also interesting in that they estimate the returns to matching for each Canadian province, which reveals considerable variation of the efficiency of matching across space. Similarly, even though Blanchard and Diamond (1989) estimated constant returns to scale in the aggregate matching function, the industry-specific matching functions exhibited economies of scale. Indirect evidence of localised returns to scale can also be found in Wheeler (2008) and Bleakley and Lin (2012).

The model is one of general equilibrium in that, like in Moretti (2011), changes in one region have a direct impact on other regions in the economy. This is in contrast to some other work, especially on urbanisation, where changes in a single city are too small to affect the rest of the country. Overall, however, we only build a partial equilbrium model because we do not explicitly analyse production and trade.

Finally, this paper attempts to contribute to the literature on labour market interactions between regions. Following Lutgen and der Linden (2015) and Beaudry et al. (2014), we also allow workers and firms from different regions to meet. In Beaudry et al. (2014), workers' migration is conditional on first forming a match with an employer; and Lutgen and der Linden (2015) entertain the possibility that the order in which workers search and move may impact on labour market outcomes. In the latter, however, the set up is such that the order does not matter, while in Beaudry et al. (2014) migration shocks are assumed to arrive sufficiently frequently as not to compromise the agglomeration process (and give rise to regional differences in utility). As a result, these models are similar to the rest of the existing literature that merges NEG and frictional labour markets in that, while providing explanations for regional variation in wages and unemployment, it is the agglomeration forces that determine regional labour markets, rather than the other way round (Epifani and Gancia (2005), Francis (2009), Zierahn (2013)).

That is not the case in this paper. We explicitly model the interaction between firms and workers across different regions, and specify the model so that for certain parameter values the "search first, move second" option is the only way for workers to migrate. The implication is that the workings of the labour market(s) has a direct impact on the pace and nature of regional divergence. Agglomeration can only happen when *both* workers and firms meet in a particular region, so when this process does not work smoothly, agglomeration itself will be impacted. To our best knowledge, this is the first model to analyse this possibility.

4 Local labour markets under autarky

We start by augmenting a simple search and match model with homogenous workers and firms a la Pissarides (2000). We make two changes: we assume that the matching technology has increasing returns to scale; and we allow the number of workers to vary.

At this stage of our analysis, we keep the variation in population size exogenous. The total population of the two regions in our model is normalised to 2; we assume that workers in our model are infinitely lived, with no births or deaths, and no migration between the two regions (autarky). This means that any change in relative population size between the two regions arises due to factors external to the model.

The aim of this section is to model how changes in population in one region relative to another shapes the differences in regional labour markets. We first summarise the set-up of the model, taken from Pissarides (2000). We then describe the equilibrium and prove it existence. The last part of this section presents some numerical simulations of the model for different levels of population.

4.1 The model

The model presented in this paper is based on a simple model of frictional labour market as introduced by Pissarides (1990). In this model, workers and firms are homogenous. The presence of imperfect information and labour market frictions, however, means that they must invest time and resources into finding their match. As a result, the economy will suffer from unemployment and unfilled vacancies even in the steady state.

The frictions in the labour market are modelled by a matching function that describes the rate at which firms with vacancies and unemployed workers meet. The number of matches made each period is denoted Φ and the arguments of the matching function are the number of unemployed in the market, u, and the number of vacancies, v,: $\Phi = \Phi(u, v)$.

We assume that $\Phi(0, v) = \Phi(u, 0) = 0$ and that it is strictly increasing in both arguments. Following Ellison et al. (2013), we also assume that Φ is homothetic and quasi-concave. This allows us to separate between the scale (i.e. market size) and composition (i.e. market tightness u/v) effects. Φ can be expressed as

$$\Phi = \Phi[m(u, v)]$$

Intuitively, $m(u, v)$ captures the size (level of activity) of the market. It is strictly increasing in both arguments, concave and homogenous of degree 1. The elasticity of m with respect to u and v is $1 - \alpha$ and α, respectively.[1] The property of constant returns to scale means that these elasticities only depend on relative vacancies and unemployment, u/v, not on their absolute levels.

The matching function Φ converts the search in a market of size m into matches. Φ is strictly increasing in m, and its elasticity with respect to m is greater than 1. This means that Φ has increasing returns to scale.

$$\eta = \frac{\partial \Phi(m)}{\partial m} \frac{m}{\Phi} > 1$$

Apart from the matching technology, labour market search depends on the behaviour of workers and firms. They choose whether to search and how long to search for depending on the parameters of the labour market. Job separation is assumed to be exogenous.

Workers can be either employed, or unemployed and searching for a job. An unemployed worker will choose to accept a job offer if the expected value of being employed exceeds the value

[1]Note that the importance of m for the outcome is another way of stating that increasing returns to scale in matching break the indeterminacy between market size and the model outcomes.

of remaining unemployed. Because there are no unemployment benefits or non-labour income in our model, the value of being unemployed consists solely of the expected utility of future employment. Once employed, the worker keeps her job until an exogenous shock destroys the match. There is no on-the-job search.

The probability that a worker finds a job is equal to the number of matches in the labour market i divided by the number of unemployed, $\frac{\Phi^i}{u_i}$. The per-period value of being unemployed is thus

$$rU_i = \frac{\Phi^i}{u_i}[W_i - U_i] \tag{1}$$

where r stands for the worker's discount rate, U_i is the value of unemployment, and W_i is the value of being employed. The per-period value of being employed is

$$rW_i = w_i + \delta[U_i - W_i] \tag{2}$$

where w_i stands for wage and δ is the probability of exogenous match destruction.

Firms maximise profits. Similarly to workers, they must decide whether to post a vacancy or not. The cost of doing so is c, but once a job is filled, it will produce $y > c$ units of output until the match is exogenously destroyed. We assume that each firm can create at most one vacancy, but the constant returns to scale in firms' production function means that the boundaries of firms do not really matter.

We assume free entry and exit from the market, which means that the value of the marginal vacancy will be driven down to 0 at all times. The cost of posting a vacancy will be just outweighed by the discounted stream of profits, times the probability of filling a vacancy, Φ^i/v^i. The resulting job-creation curve that describes firm's behaviour is

$$c = \frac{\Phi^i}{v^i}\frac{y - w_i}{r + \delta} \tag{3}$$

Because all firms and workers are the same, and each match produces the same output y, we will restrict our attention to equilibria where all the job offers are homogenous, too, and hence all will be accepted. The wage of these matches will be set by Nash bargaining, where each party receives a share of the match surplus that corresponds to their bargaining power. The problem can be expressed as

$$w_i = argmax(W_i - U_i)^{\beta}(J_i)^{(1-\beta)}$$

where β and $1 - \beta$ denote the bargaining power of workers and employers, respectively, and $J_i = \frac{y - w_i}{r + \delta}$, i.e. the value of a filled job. Using equation (2) to substitute in for the value of employment, the solution to the bargaining problem is the following wage:

$$w_i = \beta y + (1 - \beta)rU_i \tag{4}$$

The worker will be remunerated for her outside option, rU_i, and will receive a share of the net surplus $y - rU_i$ corresponding to her bargaining power β.

Plugging in the solution for wages into the equations describing worker and firm behaviour, and simplifying the former, give us the following two equations:

$$rU_i = \frac{\Phi^i}{u_i}\frac{\beta}{r + \delta}(y - rU_i) \tag{5}$$

$$c = \frac{\Phi^i}{v^i}\frac{1 - \beta}{r + \delta}(y - rU_i) \tag{6}$$

The first one gives us worker's search strategy. The second one describes firm's job creation. Together, they capture the behaviour in the labour market.

4.2 The autarkic equilibrium

Equations (5) and (6) describe the optimal behaviour of workers and firms under a steady state. For the values of their choices to stay constant, the flows in and out of unemployment must balance out: the number of new matches must just equal the number of destroyed matches:

$$\Phi^i = \delta(P_i - u_i) \tag{7}$$

P_i denotes the population size of labour market i, so $P_i - u_i$ is the number of employed workers in this region.

Equation (7) closes our description of the equilibrium. There are 3 endogenous variables: the number of unemployed u_i, the number of vacancies v_i, and worker's reservation value U_i. In equilibrium, they will satisfy the following set of 3 simultaneous equations:

$$\begin{aligned} rU_i &= \frac{\Phi[m(u_i, v^i)]}{u_i} \frac{\beta}{r+\delta}(y - rU_i) \\ c &= \frac{\Phi[m(u_i, v^i)]}{v^i} \frac{1-\beta}{r+\delta}(y - rU_i) \\ \Phi[m(u_i, v^i)] &= \delta(P_i - u_i) \end{aligned}$$

4.2.1 Existence and uniqueness

Models with increasing returns to scale may generate multiple equilibria, or none at all. In this model, we will restrict our attention to versions of the matching function that generates two equilibrium, one with no market activity ($m = 0$) and another one where $m > 0$, for each level of population P (the comparative statics for different levels of P will be explored in the next section). Our model generates two equilibria because for simplicity we assume globally increasing returns to scale (i.e. economies of scale at any level of market activity m). We then impose a restrictive assumption on the effect of vacancy creation on matches, $\alpha\eta < 1$, to ensure the existence and uniqueness of equilibrium at positive levels of market activity. The conditions for existence and uniqueness of equilibria are summarised in the following proposition; its proof can be found in the appendix. The proposition and its proof follow Ellison et al. (2014).

Proposition 1. *Assume globally increasing returns to scale in matching: $\eta(0) > 1$. Then an equilibrium (u_i^*, v^i*, U_i^*) exists for zero activity, $m = u_i^* = v^i* = U_i^* = 0$. If $\alpha\eta < 1$, there is also a second equilibrium where $m > 0$. This equilibrium will be unique for all $m > 0$.*

When $\alpha\eta > 1$ instead, there will be only one equilibrium, at zero market activity ($m = 0$).

4.2.2 Equilibrium and population size

As shown in the previous section, in the interval $\eta\alpha < 1$ the local labour market has a unique equilibrium. This means that there is a unique one-to-one mapping between population size and labour market outcomes. This section explores how different levels of population in a region drives reservation value, the number of unemployed, and the number of vacancies.

Population derivatives of the three equilibrium equations (5) - (7) reveal that all three endogenous variables are increasing functions of population size (for a full derivation, see Appendix).

$$\frac{\mathrm{d}u_i}{\mathrm{d}P} = \frac{\delta u_i}{\eta\Phi^i + \delta u_i} > 0$$

$$\frac{\mathrm{d}v^i}{\mathrm{d}P} = \frac{\delta v^i}{\eta\Phi^i + \delta u_i} > 0$$

$$\frac{\mathrm{d}U_i}{\mathrm{d}P} = \frac{y - rU_i}{r} \frac{\delta}{\eta\Phi^i + \delta u_i}[\eta - 1] > 0$$

Thanks to the increasing returns to scale in matching function, a labour market with more workers will create proportionally more matches. This will make vacancy-creation more profitable for firms, and the value of being unemployed will increase for the workers. The number of unemployed will increase for two reasons: first, mechanically, greater population entails more employed as well as unemployed; secondly, higher value of unemployment rU will translate into higher wages that workers are willing to wait longer for. However, as we demonstrate in the next section, unemployment rate will fall as population increases.

4.3 Wage curve

The wage curve is an empirical "law" documented by Blanchflower and Oswald (1995) which states that in a cross-section of regions, wages are negatively correlated with unemployment rates: regions with high wages experience only low unemployment rates, while regions with mediocre pay suffer from high unemployment. There have been various attempts at providing a theoretical underpinning for this relationship. In particular, Beaudry et al. (2014) and Epifani and Gancia (2005) explain how wages are higher in the more agglomerated region because the greater job creation there increases workers' bargaining power. A similar mechanism operates in this model, the difference being in the source of agglomeration.

Our paper can replicate this relationship when we vary relative population size of regions. As a result of increasing returns in matching, the more populated region will enjoy higher wages and lower unemployment rates than the region with fewer workers and thus smaller labour market. To make the analysis more transparent, we will refer to the bigger region as region A and the smaller one as region B.

We prove this in two steps. First, we demonstrate that wages grow in P.

$$w_A > w_B \;\Leftrightarrow\; \beta y + (1-\beta)rU_A > \beta y + (1-\beta)rU_B \;\Leftrightarrow\; U_A > U_B$$

Using equation (5) to substitute:

$$U_A > U_B \;\Leftrightarrow\; \frac{\Phi^A \beta y}{u_A(r+\delta) + \Phi^A} > \frac{\Phi^B \beta y}{u_B(r+\delta) + \Phi^B}$$

which will hold as long as

$$\frac{\Phi^A}{u_A} > \frac{\Phi^B}{u_B}$$

Intuitively, wages will be higher in the region with greater job-finding probability, since that is the only thing that differentiates the labour markets in our model.

The derivative of Φ^i/u_i with respect to P is positive for all matching functions with increasing returns to scale:

$$\begin{aligned}\frac{\mathrm{d}\frac{\Phi^i}{u_i}}{\mathrm{d}P} &= \frac{\Phi^i}{u_i^2}\frac{\mathrm{d}u}{\mathrm{d}P}(\eta - 1) \\ &= \frac{\Phi^i}{u_i}\frac{\delta}{\eta\Phi^i + \delta u_i}(\eta - 1) > 0\end{aligned}$$

More populous labour markets will indeed pay higher wages.

We now turn our attention to unemployment rate. In the absence of labour participation margin, unemployment rate is simply u_i/P_i. Its derivative with respect to P is:

$$\begin{aligned}\frac{\mathrm{d}u/P}{\mathrm{d}P} &= \frac{1}{P^2}\left[\frac{\mathrm{d}u}{\mathrm{d}P}P - u\frac{\mathrm{d}P}{\mathrm{d}P}\right]\\ &= \frac{1}{P^2}\left[\frac{\delta u}{\eta\Phi + \delta u}P - u\right]\\ &= \frac{u}{P^2}\left[\frac{\delta}{\eta\Phi + \delta u}P - 1\right]\\ &= \frac{u}{P^2}\frac{\delta P - \eta\Phi - \delta u}{\eta\Phi + \delta u}\end{aligned}$$

Using the equilibrium expression for unemployment, equation (7), we can simplify this to:

$$\frac{\mathrm{d}u/P}{\mathrm{d}P} = \frac{u}{P^2}\frac{\Phi(1-\eta)}{\eta\Phi + \delta u} < 0$$

Unemployment rate falls with population size for all labour markets with increasing returns to matching. Together with the population-driven increase in wages, our model can replicate the wage curve relationship as a consequence of regional labour markets with varying population size and increasing returns to scale in matching.

4.4 Numerical simulations

In this section, we evaluate our model for specific parameter values and matching function. This allows us to visualise the model outcomes and will serve as a basis for our simulations when we allow population to be endogenous.

We assume that the matching function Φ is a simple Cobb-Douglas function with constant returns to each argument.

$$\Phi^i = A[(u_i v^i)^\alpha]^\eta$$

To calibrate our parameters, we follow Ellison et al. (2013) (table 4).

Table 4: Calibration of parameters

y	c	r	δ	β	A	η	α
10	1	0.05	0.1	0.5	0.1	1.3	0.5

We again look at the case of two regions, the smaller B and the bigger A. We estimate the model for various levels of population such that the total population always sums up to 2. The results are summarised in Figure 12, and three selected cases are also presented in Table 5 below.

As predicted, higher population in region A translates into higher value of unemployment (which leads to higher local wages), lower unemployment rate, and higher rate of vacancies per unemployed. Region B, on the other hand, suffers from a thin labour market. Even though the actual number of unemployed is lower, so is the number of vacancies, which leads to low wages and high unemployment rate. In each of the two asymmetric cases (when population in regions is not equalised), the model generates a wage curve. The disparity in unemployment rates and wages between the two regions is greater when the population differences are greater.

In this particular case of Cobb-Douglas matching function, the model displays strong asymmetry: the improvement in labour market outcomes in region A is smaller than the decline in economic fortunes in region B.

Table 5: Equilibria for various values of population size

variable	region A			region B		
population	1	1.5	1.9	1	0.5	0.1
unemployed	0.3682	0.5	0.5973	0.3682	0.2176	0.0620
unemployment rate	37%	33%	31%	37%	43.5%	62%
vacancies	1.3397	2	2.5144	1.3397	0.6571	0.1051
market tightness	3.6	4	4.2	3.6	3	1.7
value of unemployment	72.7621	80	84.1866	72.7621	60.4024	33.8748

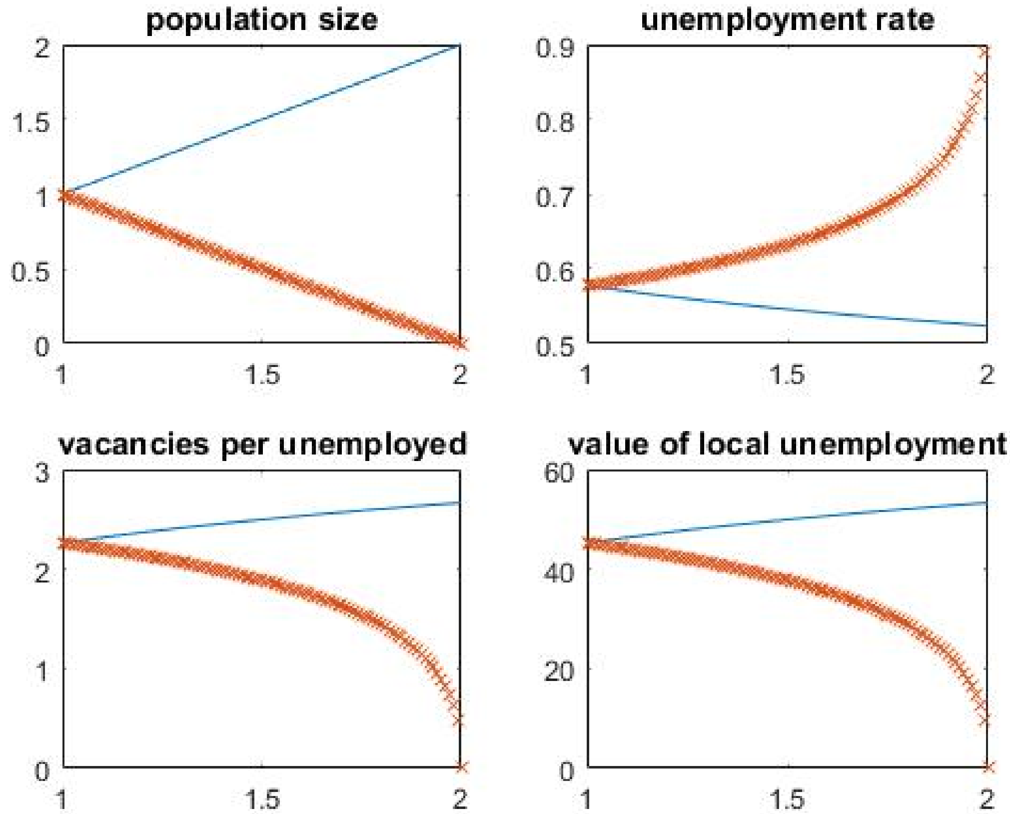


Figure 12: Equilibria for various values of population size. Region A: blue solid line. Region B: red crosses. X-axis: population in region A. Y-axes: the value of the variable on the graph.

5 Migration

In this section we allow firms and workers to move between regions. As this model focuses on static equilibria only, we model the location decisions assuming the economy is in a steady-state equilibrium. We explore whether that is indeed the case in section 6.

Firms can move costlessly and without any frictions. They can choose to locate their production in one region and hire from another, or conduct all their activity in one region only. The only limit on firm's behaviour is that they cannot move their production to a different region while they are in a match.

Workers can also move freely between regions, but migration is costly. First, there is the fixed cost of moving K that has to be paid every time a worker re-locates. This captures the cost of arranging new housing and abandoning one's social network, as well as the actual cost of hiring a moving van, etc. The second cost reflects the lack of local labour market knowledge when searching for a job. A newcomer has to pay a search cost b every period until she manages to find a job in the new labour market, after which she becomes a local and will not face this extra cost of search the next time she is on the job market.[2] [3] Only the unemployed can move.

The actual migration decisions will depend on the relative costs of migration and the net utility of moving into a different labour market. Workers' willingness to move will in turn influence firms' location and hiring decisions. In particular, the possibility of producing in one region but hiring from another opens up a second migration route. An unemployed worker may choose to relocate to a different region and search there ("move first, search later" option). Alternatively, she may be offered to move to the other region for a specific job ("search first, move later"). For a worker, the advantage of cross-regional hiring is that she can move to a new region without having to pay the extra cost of searching in an unknown labour market. This is because a cross-regional hire does not need to search in the non-domestic labour market, so that they only need to pay the fixed cost of moving K. When this job becomes exogenously terminated, the worker automatically becomes a local and does not have to pay b in her search: her professional network and letters of reference make up for the lack of actual job search experience in the non-domestic region.

From the perspective of a firm, cross-regional hiring may allow them to hire cheaper labour than in their domestic labour market (this will depend on wages in the two regions and the fixed cost of moving K). However, a firm does not distinguish between locals and migrants within a particular labour market, because the extra search costs b are sunk by the time a worker manages to meet a firm.

We assume that workers' and firms' decisions are sequenced as follows:

1. workers learn about unemployment rates and wage distributions in both regions
2. workers choose the optimal region to reside in as unemployed, and move there if this is not their home region
3. firms make decisions on whether to hire and in which region
4. firms post vacancies
5. matching happens; workers meet firms
6. potential employers and employees bargain over job surplus
7. workers migrate, if necessary for the job
8. production

[2]One can interpret this extras search cost as the cost of gaining experience in navigating a new labour market. Alternatively, Bheim and Taylor (2002) document the tax incentives for employer-driven migration in the UK.

[3]We need both types of migration costs to make the two types of migration optimal. If we only used K, there would be no need for cross-regional migration because workers would not gain by moving for a specific job. If there was only b, there would be no partial divergence equilibrium and the paper would struggle with replicating persistent regional differences.

5.1 Definition of regional labour market

What is a regional labour market? The answer to this question determines the inputs into the matching function. Here we assume a regional labour market consists of all the unemployed residing in this region (we assume the geographic borders are determined exogenously), and all vacancies *posted* there. A vacancy advertised in one region may not necessarily be located in that region. A firm in region A may decide to hire workers in region B to come and work in region A: such a vacancy would be posted in region B, and would thus be a part of region's B labour market. We denote such vacancies as v_A^B, while a "local" vacancy for a local job in region A would be v_A^A (the superscript signals where the vacancy is advertised, the subscript denotes the location of the job). The labour market in region A will thus consist of:

- all unemployed workers residing in this region, u_A
- all vacancies advertised in this region, v^A, which is the sum of vacancies for jobs located in region A, v_A^A, or in region B, v_B^A

The corresponding matching function (for region A) is:

$$\phi^A = \Phi(u_A, v^A) = \Phi(u_A, v_A^A + v_B^A)$$

The expressions for region B will be analogous. Even though the functional forms of the matching functions in the two regions are identical, the efficiency of matching may differ across regions if the size of the labour market differs. This is a direct consequence of economies of scale in Φ.

The matching function does not distinguish between vacancies with different job location. A firm located in region B has the same chances to hire a worker in region A as a firm local to region A. This is equivalent to saying that the probability of a particular geographical match (local or non-local) is proportional to the number of vacancies of the particular type.

This gives us the following job-finding probabilities:

- $\frac{\Phi_A^A}{u_A} = \frac{\Phi^A}{u_A}\frac{v_A^A}{v^A}$: the probability that a worker in region A finds a job located in region A
- $\frac{\Phi_B^A}{u_A} = \frac{\Phi^A}{u_A}\frac{v_B^A}{v^A}$: the probability that a worker in region A finds a job located in region B
- $\frac{\Phi_A^B}{u_B} = \frac{\Phi^B}{u_B}\frac{v_A^B}{v^B}$: conversely, the probability that a worker in region B finds a job located in region A

Turning to vacancy-filling probabilities, here we also must distinguish between the location of the job and where the vacancy was posted.

- $\frac{\Phi_A^A}{v_A^A} = \Phi^A \frac{v_A^A}{v^A}\frac{1}{v_A^A} = \frac{\Phi^A}{v^A}$: the probability that a firm in region A will fill its vacancy posted in region A
- $\frac{\Phi_A^B}{v_A^B} = \Phi^B \frac{v_A^B}{v^B}\frac{1}{v_A^B} = \frac{\Phi^B}{v^B}$: the probability that a firm producing in region A will fill its vacancy posted in region B
- $\frac{\Phi_B^A}{v_B^A} = \Phi^A \frac{v_B^A}{v^A}\frac{1}{v_B^A} = \frac{\Phi^A}{v^A}$: the probability that a firm producing in region B fills its vacancy posted in region A

We can see that the vacancy-filling probabilities depend on what regional labour market the vacancy is posted, not on where it is located. A firms from region B faces the same odds as a firm in region A when they post in the same labour market.

5.2 Workers

Workers now have to make two decisions: which labour market to reside in, and whether to accept a job offer. For clarity of explanation, we will focus on worker in region B, although the regions are identical in all but population levels.

As a utility maximiser, an unemployed worker living in region B will decide to migrate to region A only if the value of being unemployed there is greater than the value fo unemployment in her home region, adjusted for migration costs. The optimisation choice is:

$$\max[U_B^L, U_A^M - K]$$

where U_B^L corresponds to the value of being unemployed in home region B, U_A^M denotes the value of being an unemployed migrant in region B, and K is the aforementioned one-off moving cost. Because workers possess perfect information about wages and job-finding probabilities across regions, they will always reside in the region that gives them highest net utility.

The unemployed then decides which potential job offers to accept. If she stays in her domestic labour market B, she will agree to form a match with a firm from region A and move there if the value of being a migrant worker in that region exceeds the value of being unemployed in her domestic region, less the one-off migration cost K. The choice can be summarised as

$$\max[U_B^L, W_A^M - K]$$

where W_A^M stands for the value of being an employed migrant in region A.

She may also be offered a local job offer, the value of which (for region B) is W_B^L. In an equilibrium, the value of being employed will always exceed the value of being unemployed, which is why she will always accept such a vacancy if it is offered to her.

To summarise, an unemployed worker in (for example) region B faces three possible scenarios. First, she can move to region A to search for jobs there. This will give her net utility of $U_A^M - K$. If she chooses to stay in her home region and search there, she will receive utility of U_B^L until she decides to accept a job offer. If she accepts a job located in region A and move to A as a migrant worker, the net utility she will receive is $W_A^M - K$. The third option is to accept a local job, which gives the utility of W_B^L.

A worker's behaviour can be summarised in the following 8 Bellman equations:

$$rU_A^L = \frac{\Phi_A^A}{u_A}[W_A^L - U_A^L] + \frac{\Phi_B^A}{u_A}[W_B^M - U_A^L - K] \tag{8}$$

$$rU_B^L = \frac{\Phi_B^B}{u_B}[W_B^L - U_B^L] + \frac{\Phi_A^B}{u_B}[W_A^M - U_B^L - K] \tag{9}$$

$$rU_A^M = -b + \frac{\Phi_A^A}{u_A}[W_A^L - U_A^M] + \frac{\Phi_B^A}{u_A}[W_B^M - U_A^M - K] \tag{10}$$

$$rU_B^M = -b + \frac{\Phi_B^B}{u_B}[W_B^L - U_B^M] + \frac{\Phi_A^B}{u_B}[W_A^M - U_B^M - K] \tag{11}$$

$$rW_A^L = w_A^L + \delta[U_A^L - W_A^L] \tag{12}$$

$$rW_B^L = w_B^L + \delta[U_B^L - W_B^L] \tag{13}$$

$$rW_A^M = w_A^M + \delta[U_A^L - W_A^M] \tag{14}$$

$$rW_B^M = w_B^M + \delta[U_B^L - W_B^M] \tag{15}$$

Expressions (8) - (11) are reservation values for different scenarios. They reflect the fact that there are two different types of employment available - either local or in a different region. Notice that to take a non-domestic job, we have to adjust the net utility gain for the one-off migration cost K. Equations (8) and (9) capture value of being unemployed as a local in regions A and B, respectively. Equations (10) and (11) correspond to being an unemployed migrant, and hence are augmented by the extra cost of search b. w_i^L stands for wages of a local hire, while w_i^M denotes the wage of a migrant worker (i.e. wage paid in a cross-regional match). Equations (12) - (15) correspond to values of being employed in these 4 scenarios.

5.3 Firms

Similarly to workers, firms also have to make two optimisation decisions: where to locate their production, and which labour market to hire from. These two choices are separate because while the vacancy-filing probabilities depend on the market where the vacancy was posted, the wage a firm has to pay also depends on which market production takes place. This is not because living or housing costs differ between regions - we assume both our regions are identical in that respect - but because various labour market sizes correspond to different job opportunities.

Firms in our model are allowed to post vacancies in one region at a time. This is to prevent a situation in which a worker is bargaining with two employers at the same time.

In choosing which market to settle in, firms compare regional values of vacancies. While the per-period cost of posting a vacancy, c, is the same everywhere, the probability a vacancy will be filled, and the wage will end up paying to its workers, does differ. In region A, the value of local vacancy is:

$$rV_A^L = -c + \frac{\Phi^A}{v^A}\frac{y - w_A^L}{r+\delta}$$

while the value of posting a vacancy in region B for a firm producing in region A is:

$$rV_A^M = -c + \frac{\Phi^B}{v^B}\frac{y - w_A^M}{r+\delta}$$

A firm will always choose the highest of these four values. Given the free entry and exit into the market, as well as between regions, the value of the marginal vacancy will be driven to 0; in this process, the value of some vacancies may become negative (which means no firm will choose to create them). This also implies that the marginal firm will be indifferent between settling in region A and B. Taking this into account, the four Bellman equations above can be re-written as more conventional job creation curves:

$$c \geq \frac{\Phi_A}{v^A}\frac{y - w_A^L}{r+\delta} \tag{16}$$

$$c \geq \frac{\Phi_B}{v^B}\frac{y - w_B^L}{r+\delta} \tag{17}$$

$$c \geq \frac{\Phi_B}{v^B}\frac{y - w_A^M}{r+\delta} \tag{18}$$

$$c \geq \frac{\Phi_A}{v^A}\frac{y - w_B^M}{r+\delta} \tag{19}$$

which will hold with equality if those vacancies are being created.

5.4 Wages

Interactions between regional labour markets give rise to wage heterogeneity. When no migration is possible, there are only two wages - one for each regional labour market. When cross-regional hiring is possible, however, the model produces 4 different wages, corresponding to the 4 different geographical combinations between workers and firms:

- w_A^L: the wage of local workers in region A
- w_A^M: the wage of migrant workers in region A
- w_B^L: the wage of local workers in region B
- w_B^M: the wage of migrant workers in region B

Because the bargaining process is repeated every period, and because of the homogeneity assumption, all wages within a particular geographical match will be the same.

Wages of local hires will be derived in the same way[4] as in the model without migration, because the form of workers' outside option is still the same: U_i^L. Even though there are de facto two different types of workers applying for these jobs, locals and migrants, firms do not discriminate between them. This is because by managing to find a vacancy, a migrant worker has gathered sufficient labour market experience to become a local and not to have to pay the extra cost of search b in future search. From the perspective of a firm, then, the costs of migration are sunk by the time wage bargaining begins, and locals and migrants will have the same outside options. Hence, the local wages in region A and B respectively are:

$$w_A^L = \beta y + (1-\beta) r U_A^L \tag{20}$$

$$w_B^L = \beta y + (1-\beta) r U_B^L \tag{21}$$

Wages of cross-regional hires will be different. Firstly, the worker would not migrate in the absence of this job offer, so K enters worker's outside option. Second, cross-regional hires only become locals in the new region upon exogenous termination of the match. Until then, their outside option is that of an unemployed migrant, U_i^M, not of local unemployed, U_i^L. The rationale for this is that as someone without actual search experience in the new labour market, they would normally rely on a reference letter or their professional network to find a new job. They cannot access this if they end the match endogenously.

The wage of a migrant worker is thus solution to the following Nash bargaining problem:

$$w_A^M = argmax(W_A^M - U_A^M - K)^\beta (J_A^M)^{(1-\beta)}$$

$$\text{F.O.C.: } W_A^M - U_A^M - K = \beta(J_A^M + W_A^M - U_A^M - K)$$

$$\frac{w_A^M + \delta U_A^L}{r+\delta} - U_A^M - K = \beta\left(\frac{y - w_A^M}{r+\delta} + \frac{w_A^M + \delta U_A^L}{r+\delta} - U_A^M - K\right)$$

$$w_A^M = \beta y + (1-\beta)(r+\delta)(U_A^M + K) - (1-\beta)\delta U_A^L \tag{22}$$

The wage equation attributes the worker her share of output, βy, and remunerates her for her outside option, $U_A^M + K$. However, this outside option is adjusted for the outside option the cross-regional hire is going to gain once the current match is destroyed and she resides in region

[4]This does not mean that local wages are not affected by the possibility to migrate one way or another. Migration increases workers' utility because it expands their outside options. This will be reflected in the value of unemployment.

A as a local (U_A^L). If the value of living in A in greater than in B, this adjustment will be negative: the employed is able to pay the worker less because a part of the remuneration is the move to a more prosperous region.

Wage for migrant worker in region B can be derived analogously:

$$w_B^M = \beta y + (1-\beta)(r+\delta)(U_B^M + K) - (1-\beta)\delta U_B^L \tag{23}$$

5.5 Cross-regional hiring

Cross-regional hiring arises when a firm decides to settle its production in one region but hire workers from the other region. In this section, we will show that if this type of hiring will only work in one direction: firms will hire workers from the smaller region to come to work in the larger region. Cross-regional hiring in the opposite direction (from A to B) is not profitable. This is a general result of the model and holds true for all possible equilibria.

Proposition 2. *If cross-regional hiring occurs (i.e. it is optimal and profitable), it will happen in only one direction, that of from less populous to more populous regions. It will never be profitable and optimal to hire workers from the more populous region to work in jobs in the region with smaller population.*

Proof. We will prove this proposition by contradiction. We will examine the conditions that need to hold for two-way cross-regional hiring to exist, and then prove they can never hold. Throughout the proof, we will assume that region A is greater (and thus has a thicker labour market) than region B. Because our model is symmetric, this assumption only determines *which* of the two directions of cross-regional hiring might exist.

We start by looking at firms. For two-way cross-regional hiring to exist, the 4 job-creation curves captured in (16)-(19) must all hold with equality. Only when the values of all 4 types of vacancies is non-negative the firms will create all 4 types of jobs. (They cannot be positive because of our arbitrage assumption.) Assuming one region (A) is greater than the other (B), this yields a set of relationships between regional wages, condition (24).

$$c = \frac{\Phi_A}{v^A}\frac{y - w_A^L}{r+\delta} = \frac{\Phi_B}{v^B}\frac{y - w_B^L}{r+\delta} = \frac{\Phi_B}{v^B}\frac{y - w_A^M}{r+\delta} = \frac{\Phi_A}{v^A}\frac{y - w_B^M}{r+\delta}$$

$$\therefore \text{ if } \frac{\Phi_A}{v^A} > \frac{\Phi_B}{v^B} \text{ then } \Leftrightarrow w_A^L = w_B^M > w_B^L = w_A^M \tag{24}$$

The first part, the equality between migrant wages in region B and local wages in A, ensures that hiring in labour market in region A is equally profitable for all participants. The second part, inequality between local wages in the two regions, is a condition for local hiring to exist at the same time: if a firm is to create vacancies in the thinner labour market in region B, it needs to be compensated by lower wages there. Thirdly, migrant wages in region A must be below local wages in this region. This is because to bring in a migrant worker, the firm has to fill a vacancy in the thinner (and hence more expensive) matching process, and so needs to be compensated by keeping greater part of output. The opposite holds for the relationship between migrant and local wages in region B: hiring locally in this case is the more costly option, which is why local wages need to be lower than migrant wages. This is also why migrant wages when hiring in region A (w_B^M) must be greater than when cross-regionally hiring from B (w_A^M).

From workers' perspective, for cross-hiring to exist, there must be some range of wages at which they prefer it to local jobs. A worker living in region B will accept a job in region A if its value weakly exceeds the value of staying unemployed locally, net of migration cost:

$W_A^M - K \geq U_B^L$. A sufficient condition for this inequality to hold is that the value of a migrant job (net of migration cost) is weakly greater than the value of local job. [5]

$$W_A^M - K \geq W_B^L$$

The advantage of working with W^L rather than U^L is that the former allows us to formulate workers' conditions in terms of wages. Using expressions (14) and (13) to substitute:

$$\frac{w_A^M + \delta U_A^L}{r+\delta} - K \geq \frac{w_B^L + \delta U_B^L}{r+\delta}$$

$$\delta(U_A^L - U_B^L) - (r+\delta)K \geq w_B^L - w_A^M$$

The differences in labour market efficiency (here in region A's favour) means that $U_A^L > U_B^L$. The sign of the left hand side of the inequality is thus ambiguous. A necessary condition for the above to hold is when local wages are at most the same as migrant wages:

$$w_A^M \geq w_B^L \tag{25}$$

Similarly, a worker in region A would agree to move to region B for a job only if the value of such job exceeded the value of having a job in her domestic region, which is greater than being unemployed domestically (net of migration costs):

$$W_B^M - K \geq W_A^L \geq U_A^L$$

This implies:

$$\frac{w_B^M + \delta U_B^L}{r+\delta} - K \geq \frac{w_A^L + \delta U_A^L}{r+\delta}$$

$$w_B^M - w_A^L \geq \delta(U_A^L - U_B^L) + (r+\delta)K$$

The right hand side of this inequality is positive. As a consequence, migrant wages in region B must be strictly greater than local wages in region A. Intuitively, locals living in the more vibrant market place will require compensation to move to a labour market with poorer opportunities.

$$w_B^M > w_A^L \tag{26}$$

Are these conditions (25) and (26), compatible with firms' profit maximisation, as given by condition (24)? While condition (25) can be reconciled with the firms' profitability conditions when $w_A^M = w_B^L$, the same cannot be said for condition (26). It requires firms located in B but hiring in A to pay more than local firms to induce workers to move to the thinner labour market. However, this will not be profitable: when the values of all vacancies are driven to 0, and two firms face identical hiring costs, they must pay the same wages. If firms located in region B paid higher wages than their local counterparts in region A they would make a loss. There will be no cross-regional hiring of workers residing in region A.

The flow of worker migrants is only possible from the smaller region (B) to the larger one (A). The compatibility of condition (25) with firms' cross-hiring conditions proves this. Workers value residing in A more than in B, which is why they are willing to be paid as little as w_B^L (and bear the migration cost K) to move there. At the same time, given the thin labour market in B, w_B^L is the most firms located in A can afford to pay their cross-hires without making a loss relative to domestic B firms. □

[5] Value of unemployment will always be below value of employment (for a particular geographical type). Otherwise workers would have no incentive to search, and the labour market would disappear.

Looking at the job creation inequalities (16) - (19), a consequence of this proposition is that the value of a cross-regional vacancy from A to B will be negative, while the other three job creation conditions will hold with equality.

$$c = \frac{\Phi_A}{v^A} \frac{y - w_A^L}{r + \delta}$$

$$c = \frac{\Phi_B}{v^B} \frac{y - w_B^L}{r + \delta}$$

$$c = \frac{\Phi_B}{v^B} \frac{y - w_A^M}{r + \delta}$$

$$c > \frac{\Phi_A}{v^A} \frac{y - w_B^M}{r + \delta}$$

What will happen when the two regions are of identical size? There will be no cross-regional hiring because it would be suboptimal for workers. When the two regions are identical in size, they offer the same labour market outcomes: job-finding probability, unemployment rates and wages will be equally good in both locations. In the absence of migration costs, a worker would be indifferent between accepting in either region, and cross-regional hiring would flourish - indeed, there would be no practical difference between hiring from domestic market or from another region. Having to pay K to move, however, changes this: it is no longer optimal for a worker to pay the cost of moving to get the same job (same wage at the same job-finding probability) she could get at home. Knowing this, firms have no incentive to offer any cross-regional contracts, and this market will not exist.

5.6 Migration of the unemployed

Similarly to cross-regional hiring, migration of unemployed workers will also occur only in one direction when regions are asymmetric.

Proposition 3. *If the unemployed find it optimal to move (migration of the unemployed exists), they will only migrate from the less populous region to the more populous one.*

Proof. Our proposition is equivalent to stating that if $U_A^M - K > U_B^L$ holds, then $U_B^M - K > U_A^L$ can not. The proof uses the fact that the value of being unemployed as a local U_i^L is always greater than being an unemployed migrant in the same region, U_i^M.

$$\begin{aligned}
U_A^M - K &> U_B^L \\
U_A^M &> U_B^L + K \\
U_A^L &> U_A^M > U_B^L + K \\
U_A^L &> U_A^M > U_B^L + K > U_B^L - K \\
U_A^L &> U_A^M > U_B^L + K > U_B^L - K > U_B^M - K \\
U_A^L &> U_B^M - K
\end{aligned}$$

The last inequality is a direct contradiction of the condition for unemployed migration from region A to B. □

6 Steady state when population is endogenous

In section 3, we defined the three equations that describe labour market equilibria when population is exogenous. We showed that at a particular interval of increasing returns to matching, there will be a one-to-one mapping between population size and labour market outcomes.

In section 4, we modelled the location choice of workers and firms in terms of labour market decisions.

In this section, we allow population to be determined by migration and hiring decisions of workers and firms. We identify the population intervals at which the economy is in a steady-state equilibrium, and analyse what type of migration will be optimal outside of the steady-state equilibria. Because we do not derive the dynamics of the model, all equilibria are steady-state equilibria.

6.1 No-migration conditions

We define a steady-state equilibrium as one where population size is constant. As this is the driving factor of labour market in our model, constant regional population translates into constant endogenous variables.

In a model without births or deaths, constant relative population means no migration, either via cross-regional hiring or of unemployed workers. We therefore identify steady-state equilibria as those autarkic equilibria where migration is not optimal. In Propositions 2 and 3 we showed that if migration occurs, it will be only in one direction, from the smaller into the bigger region. A steady state will thus one where no migration occurs, i.e. it is not possible for migration flows in different directions to cancel each other out.

Migration of the unemployed depends on workers alone, but for cross-regional hiring to exist it must be optimal for both workers and firms. This gives us 3 no-migration conditions that will identify the steady-state equilibria in our model.

There is nothing in our model that would prevent the existence of multiple steady-state equilibria. As we will demonstrate below, no-migration conditions are in the form of inequalities, which is why there can be more than one value of regional population P_A for which migration will not be optimal. We define the exact steady-state equilibrium intervals in section 5.3.

No migration of the unemployed If the unemployed migrate, they will only move from the smaller region to the larger one. This will not be optimal if the net value of searching in region A is below the value of being unemployed in region B.

$$U_A^M - K < U_B^L \;\Leftrightarrow\; U_A^L - U_B^L < K + \frac{b}{r + \lambda_A^A} \qquad \text{(Condition 1)}$$

Intuitively, unemployed in region B will not migrate to region A if the extra utility is lower than the fixed cost of moving K plus the expected discounted cost of search.

No cross-regional hiring - worker condition Workers will refuse a job in region A if its value, less the migration cost, is below the value of staying unemployed in region B:

$$W_A^M - K < U_B^L \;\Leftrightarrow\; U_A^L - U_B^L < \beta K + (1-\beta)\frac{b}{r + \lambda_A^A} - \frac{r}{\lambda_A^A} U_A^L \qquad \text{(Condition 2)}$$

This condition tells us that when workers agree to a cross-regional match, the utility gain must exceed the share of total migration costs they cannot retrieve in wage bargaining. Note that the migration costs b and K are adjusted by the value of being unemployed in the larger, more prosperous region A: this is because cross-regional hiring are willing to take a relative pay cut in order to migrate to the more prosperous region.

No cross-regional hiring - firm condition Firms producing in A will not find it profitable to post vacancies in region B if the value of such vacancy is smaller than the value of hiring locally in this market. In other words, under which circumstances will a firm in A not advertise in B, given that local firms in B are hiring? This gives us the following condition where the left-hand side of the inequality corresponds to local job creation curve in B, while the right-hand side is the value of a cross-regional vacancy (which is why it is adjusted by migration costs):

$$c = \frac{\Phi_B}{v^B}\frac{1-\beta}{r+\delta}(y - rU_B^L) > \frac{\Phi_B}{v^B}\frac{1-\beta}{r+\delta}(y - rU_A^L) + \frac{\Phi_B}{v^B}(1-\beta)\left[\frac{b}{r+\lambda_A^A} - K\right] \Leftrightarrow$$

$$U_A^L - U_B^L > \frac{r+\delta}{r}\left[\frac{b}{r+\lambda_A^A} - K\right] \qquad \text{(Condition 3)}$$

The right-hand side of this condition captures the difference between extra search costs due to migration, and the fixed cost of migration. When this difference is greater than the utility gain of moving (left-hand side), the extra search costs are large enough to prevent unemployed workers from migrating on their own (move first, search later), and, as a consequence, this is what makes cross-regional vacancies attractive to workers. If extra costs of search are relatively small compared to the utility gain, a worker will not find it optimal to accept a wage cut in order to move to the more prosperous region, and firms will not find it profitable to create such vacancies.

These three conditions impose lower and upper limits on the difference in regional values of unemployment. A steady-state equilibrium will exist where the sum of migration costs is large enough to disincentivise the unemployed from migrating, but where at the same time the cost of search b is not large enough to encourage cross-regional hiring.

6.2 Types of steady-state equilibria

Each of the three conditions can either be satisfied, or not. This gives us 8 different combinations. 3 of these, summarised in table 6, will be steady-state equilibria, because neither type of migration is profitable or optimal.

Table 6: Types of steady-state equilibria

combination	**condition 1**	**condition 2**	**condition 3**	**steady state equilibrium?**
1	holds	holds	holds	yes
2	holds	holds	does not hold	yes
3	holds	does not hold	holds	yes

The other 5 combinations of the no-migration conditions will not result in a steady-state equilibrium. Table 7 summarises these. In each case, at least one type of migration will be profitable and optimal, which means that the economy is not in a steady-state equilibrium at the level of population.

A closer look at the combinations 5 and 6 reveals that these are not possible. In both 5 and 6, condition 1 does not hold but condition2 does. It turns out this is not possible. Intuitively, when the unemployed in region B find it optimal to migrate to region A (condition 1 does not hold), they would also find it optimal to accept job offers from this region (condition 2 does not hold). Hence it is not possible for condition 2 to hold when condition 1 does not. Algebraically, in these two cases condition 1 determines the lower boundary of $U_A^L - U_B^L$, and condition 2 determines the upper boundary. Given that β falls within the unit interval, the lower boundary $K + \frac{b}{r+\lambda_A^A}$ is greater than the upper boundary $\beta K + (1-\beta)\frac{b}{r+\lambda_A^A} - \frac{r}{\lambda_A^A} U_A^L$. This is why neither of these two condition combinations can arise, and we will not consider them further.

Table 7: Combinations of no-migration conditions under which no steady-state equilibrium exists

combination	**condition 1**	**condition 2**	**condition 3**	**no steady state because...**
4	holds	does not hold	does not hold	incentives for cross-regional hiring
5	does not hold	holds	holds	incentives for migration of unemployed
6	does not hold	holds	does not hold	incentives for migration of unemployed
7	does not hold	does not hold	holds	incentives for migration of unemployed
8	does not hold	does not hold	does not hold	incentives for types of migration

Intuitively, tables 6 and 7 answer the question of how would the regional labour markets react if the assumption of autarky was lifted. For a given population distribution (P_A, P_B), no-migration conditions can either hold (in one of the three combinations 1-3), or not (again in 3 different ways, 4, 7 and 8). In the former case, when there is not incentive to migrate, an autarkic equilibrium would also be a steady-state equilibrium when population becomes endogenous. In the latter case, because workers and firms find it optimal to migrate and/or hire cross-regionally, an autarkic equilibrium would not be a steady-state equilbrium when population is endogenous. Instead, depending on what type of migration is optimal, workers would move between regions until a steady-state equilibrium is reached.

6.3 Location of steady-state equilibria

As explained above, there are 3 combinations of no-migration conditions that result in a steady-state equilibrium. However, not all 3 may necessarily exist for all values of migration costs K, b.

Ideally, we would like to identify the intervals of population in region A, P_A, for which no-migration conditions will be satisfied. This would identify those equilibria from the autarkic model that are also steady states when the autarky assumption has been lifted. However, given the structure of the model, it is not clear that comprehensible closed-form analytical solutions could be found. Instead, to track the intuition of the model, we decided to keep the definition of the parameter space, and the steady-state equilibrium locations, as functions of P_A ($\lambda(P)$, the local job-finding probability in region A, and $\Delta U = U_A^L = U_B^L$).

The results are summarised in table 8. The first two columns specify the parameter space, which is given in a series of conditions for K, b. Given the particular parameter configuration, column 3 tells us which steady-state equilibrium types exist, i.e. which of the 3 combinations of no-migration conditions are possible given the relative size of the two different migration costs. Column 4 locates the steady-state equilibria. Alternatively, we can say that a steady-state equilubrium (of a particular type) exists if all the conditions outlined in columns 1, 2 and 4 hold simultaneously.

The autarkic equilibria that do not satisfy the conditions of steady state are not steady-state equilibria. The interval of their location, as a function of P_A, is a complement to the intervals of steady-state equilibria. This is summarised in table 9, which also denotes what type of migration will be optimal on each off-equilibrium interval.

This exercise allows us to make several observations.

First, a steady-state equilibrium exists when the population in the two regions is symmetric, $P_A = P_B$. At this point, the difference in utilities will be zero, which makes costly migration unattractive, regardless of the particular parameter values.

Similarly, a second steady-state equilibrium is located at the point of total divergence, i.e. $P_B = 0, P_A = 2$. We know that this will be a steady state from Propositions 2 and 3, which specify that migration will only occur from the smaller region to the bigger one.

Finally, there will be a partial divergence steady state for as long as the frictions in the labour market exist, i.e. $\lambda(P_A) < \infty$.

What can be said about stability of these multiple equilibria? As will be shown in figures 13 and 14, this depends. If we consider stability in terms of reaction to small perturbation in population size of a region, some steady-states are stable because they lie in an interval of steady states: so the perturbed population is also a steady state. The "edges" of these intervals, on the other hand, are unstable, as small disturbance in population size will start off a path of migration adjustment.

Table 8: Existence of steady-state equilbria for various parameter values

scenario	parameter values		which type of steady-state equilibrium?	interval
1	$\frac{b}{r+\lambda(P_A)} < K$	[no second parameter restriction]	only 1 and 3 exist	$\Delta U(P_A) \in \langle 0, K + \frac{b}{r+\lambda(P_A)})$
2	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} > \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right) >$ $> \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A)$	only 2 and 3 exist	$\Delta U(P_A) \in \langle 0, \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A))$ $\cup \left(\frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right), K + \frac{b}{r+\lambda(P_A)}\right)$
3	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} >$ $> \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A) >$ $> \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right)$	1, 2, 3 exist	$\Delta U(P_A) \in \langle 0, K + \frac{b}{r+\lambda(P_A)})$
4	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} < \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right)$	only 2 exists	$\Delta U(P_A) \in \langle 0, \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A))$

Table 9: Existence of off-equilibrium population adjustment for various parameter values

scenario	parameter values		which type of off-equilibrium path?	interval
1	$\frac{b}{r+\lambda(P_A)} < K$		only migration of unemployed (7)	$\Delta U(P_A) \in (K + \frac{b}{r+\lambda(P_A)}, \infty)$
2	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} > \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right) >$ $> \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A)$	cross-regional hiring (4) [1st interval] migration of unemployed (7) [2nd interval]	$\Delta U(P_A) \in$ $(\beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A), \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right))$ $\Delta U(P_A) \in (K + \frac{b}{r+\lambda(P_A)}, \infty)$
3	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} >$ $> \beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A) >$ $> \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right)$	only migration of unemployed (7)	$\Delta U(P_A) \in (K + \frac{b}{r+\lambda(P_A)}, \infty)$
4	$\frac{b}{r+\lambda(P_A)} > K$	$K + \frac{b}{r+\lambda(P_A)} < \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right)$	cross-regional hiring only (4) [1st interval] both types of migration (8) [2nd interval] only migration of unemployed (7) [3rd interval]	$\Delta U(P_A) \in$ $(\beta K + (1-\beta)\frac{b}{r+\lambda(P_A)} - \frac{r}{\lambda(P_A)}U_A^L(P_A)), K + \frac{b}{r+\lambda(P_A)})$ $\Delta U(P_A) \in (K + \frac{b}{r+\lambda(P_A)}, \frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right))$ $\Delta U(P_A) \in (\frac{r+\delta}{r}\left(\frac{b}{r+\lambda(P_A)} - K\right), \infty)$

Table 10: Steady-state equilibrium intervals for various migration costs (numerical simulation of table 8)

scenario	K	b	interval
1	24	5	$P_A \in [1, 1.85)$
2	15	6	$P_A \in [1, 1.0065) \cup (1.62, 1.79)$
3	25	6	$P_A \in [1, 1.88)$
4	14	8	$P_A \in [1, 1.11)$

Table 11: Off-equilibrium intervals for various migration costs (numerical simulation of table 9)

scenario	K	b	interval
1	24	5	$P_A \in [1.85, \infty)$
2	15	6	$P_A \in [1.0065, 1.62) \cup (1.79, \infty)$
3	8.5	2	$P_A \in [1.88, \infty)$
4	3.5	2	$P_A \in [1.11, 1.86) \cup [1.86, 1.88] \cup (1.88, \infty)$

7 Modelling regional differences and divergence

How can our model explain the stylised facts of regional disparities?

In section 4.4, we already demonstrated that the combination of labour market frictions and increasing returns to matching can replicate the qualitative differences in regional wages and unemployment rates, as well as the particular wage-curve pattern. Figure 11 demonstrated that the wage-curve relationship is linked to population size of a region: larger regions were associated with higher wages, and hence also lower unemployment. This corresponds to our model, which predicts that regions with greater population will benefit from higher wages and lower unemployment rates, because labour market frictions get smaller as the labour market increases.

To explain the persistence of labour market outcomes, and the relatively low migration between regions documented in section 2, we turn to the 4 parameter scenarios outlined in tables 8 and 9. We simulate them numerically, using the same assumption about the functional form of the matching function and calibration values as in section 4.4. We evaluate the 3 no-migration conditions for the 4 different parameter scenarios and identify the intervals of steady-state equilibria.

The results of these simulations captured in figures 13 and 14. There are 3 curves within each figure, depicting the 3 no-migration conditions. When a curve is below 0, the value of particular action is suboptimal, and the given no-migration condition holds. The solid line corresponds to condition 1 (no migration of the unemployed); the dashed line captures condition 2 about optimality of cross-regional jobs for unemployed in region B; and the dotted line plots profitability of such vacancies to firms, which is no-migration condition 1. A steady-state equilibrium is defined as the interval of P_A, population in region A, where neither type of migration is optimal, i.e. condition 1 holds and at least one of conditions 2 and 3 hold. Steady-state equilibria are marked by a circle and thick line in the diagrams in figures 13 and 14. The diagrams also report which type of the 8 no-migration conditions is relevant for given P_A interval.

Within the interval of these steady-state equilibria, a small shock to the population of either region is not going to have any positive feedback effect. We do not model dynamics of population adjustment explicitly, but because we know whether migration conditions hold or not, we can pin down the intervals of P_A at which regional population evolves with perfect path-dependence, i.e. without starting a process of divergence.

Scenario 1 is one where migration costs are fairly high. This is why there is an interval of steady-state equilibria at and around the symmetric regional labour market ($P_A = P_B = 1$). Intuitively, when the differences in labour market size between the two regions are fairly small or zero, migration will not be optimal because it is too costly compared to the small difference in inter-regional utility. When the differences get bigger, it will become optimal to pay K to move from region B, and migration of the unemployed becomes optimal. The economy stops being in an equilibrium and instead the process of divergence begins. The two regions will diverge until all population has moved to the larger region A. As the migration costs increase, the point at which this happens (i.e. condition 1 crosses the x-axis) moves to the right: the inter-regional difference in utility must get large before migration of the unemployed becomes optimal. Note that there is no cross-regional migration because fixed costs of moving K are too large relative to the extra cost of search b. This gives firms in A little advantage when hiring in region B and the dotted line corresponding to condition 3 is thus always below 0.

Scenario 3 is qualitatively similar: an initial interval of steady-state equilibrium when P_A is relatively small changes into off-equilibrium adjustment via migration of unemployed until a final asymmetric steady-state is reached where $P_A = 2$. The only difference is that unlike in scenario 1, cross-regional hiring is optimal for firms (we can see that the dotted line is above the x-axis on a small interval starting at $P_A = 1$), because b is now relatively larger relative to K. Yet, no cross-regional hiring occurs because the interval when condition 3 is positive does not overlap at all with the interval at which condition 2 is positive.

In scenario 4, there are also 2 intervals with steady-state equilibria, one when regional differences are small and another one at the point of absolute divergence, $P_A = 2$. The point at which migration starts is much closer to $P_A = 1$. We observe that migration costs in this scenario are not particularly large - in fact, both scenarios 1 and 3 have at least one dimension of the cost exceeding those in scenario 4. Adjustment begins much earlier because of the viability of cross-regional hiring. b is comparatively large, which makes cross-regional hiring attractive for firms; at the same time, K is not too big, which allows workers to take up the job offers elsewhere. When divergence drives inter-regional differences in utility even higher, the unemployed will find it optimal to migrate themselves until divergence is complete. Note that cross-regional hiring ceases to be profitable for firms at this stage (when P_A is much grater than P_B): although wages are low in region B and there is a large number of unemployed, the thinness of the local labour market makes it too costly to advertise vacancies there.

Scenario 2 is different. In addition to two steady-state equilibria - one at $P_A = 1$ and the other at $P_A = 2$ - there is a third interval at intermediate level of divergence. Migration costs make moving suboptimal when the two regions are identical in size - this is the symmetric steady-state equilibrium. As K is not too large to prevent workers from accepting cross-regional jobs, and as the relatively large b makes creating such jobs profitable for employers, adjustment via cross-regional hiring will occur if the initial symmetric steady-state equilibrium is disturbed. The two regions diverge as workers are poached to work in region A. However, as labour market in B gets less and less efficient, hiring there becomes more and more expensive, and firms in A will not be able to compete with local firms in the market when it comes to hiring. Migration of workers comes to a halt, but the sum of migration costs is still too large to justify migration of the unemployed. This is what gives rise to the second steady-state equilibrium interval. When regional differences increase above the cost of both migration costs, the unemployed will find it optimal to move, and the regions will completely diverge.

These 4 scenarios give two explanations for persistence in large regional differences coupled with relatively low migration rates. It may be the case that the one-off cost of migration is too large to incentivise movement of unemployed from peripheral regions to the economic core, but the extra cost of searching in a new labour market are too small to make cross-regional hiring worthwhile for employers in the larger region. Or it may be the case that these costs *are* quite large, but the smaller labour markets got too thin to effectively match employers in the more prosperous regions with unemployed living in the less prosperous areas. In that situation, the local labour market does not work well enough to allow workers to move, and they cannot do so themselves because of the high fixed costs of moving. Either way, labour market frictions in junction with costs of migration would prevent agglomeration to run its full course, locking workers in geographical pockets of low utility.

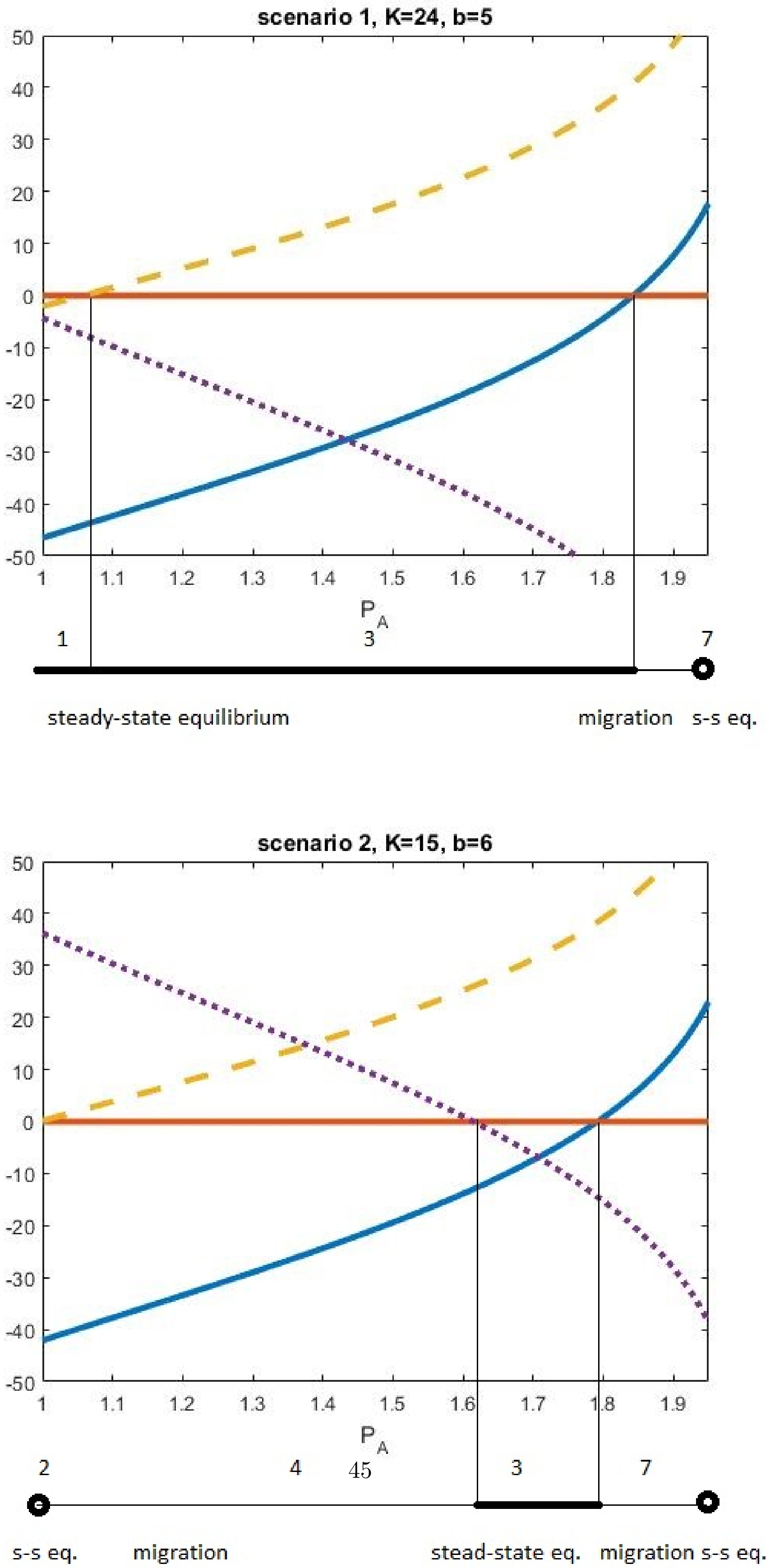
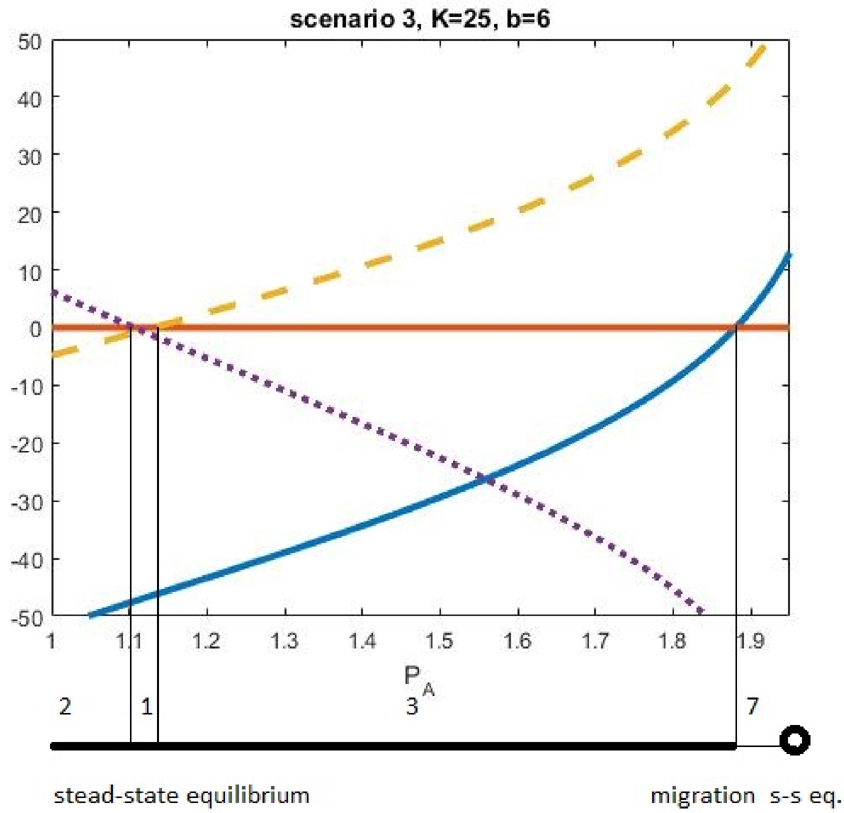


Figure 13: Scenarios 1 and 2. Bold line and circle: steady-state equilibria. Numbers refer to relevant combination of no-migration conditions.



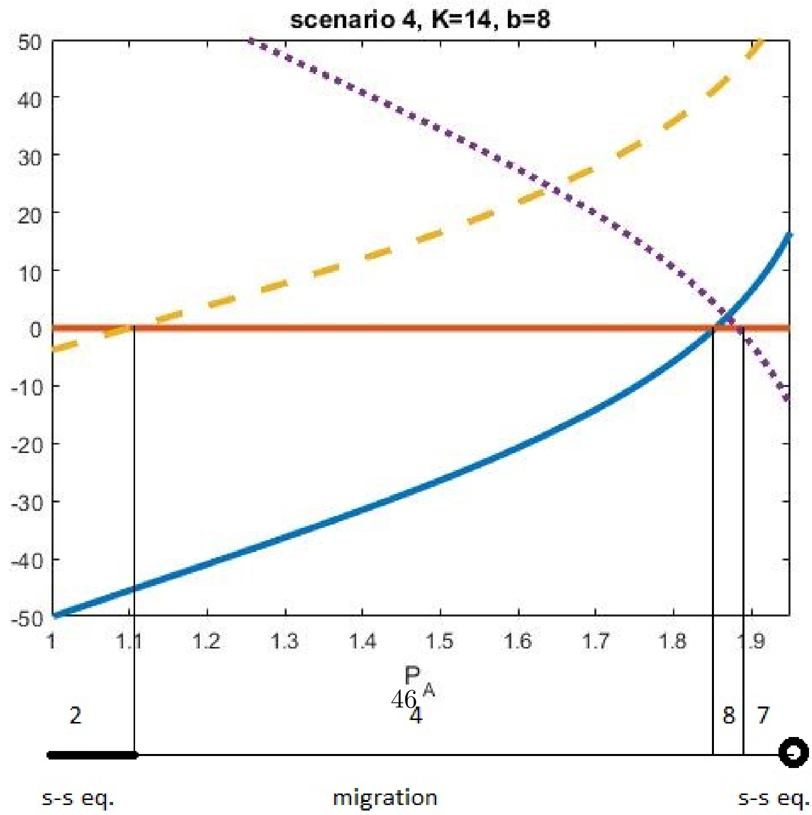


Figure 14: Scenarios 3 and 4. Bold line and circle: steady-state equilibria. Numbers refer to relevant combination of no-migration conditions.

8 Conclusion

The aim of this paper was to explore the role of labour market frictions and costs in determining relative regional outcomes. We have shown that the combination of increasing returns to matching and migration costs can generate persistent regional variation in wages and unemployment rates. We have also documented how migration costs determine the pattern of regional divergence.

A crucial outcome of the model is that firms may not choose to take advantage of the lower wages and relatively larger pool of unemployed in the periphery region, despite the fact that they do not face any migration or transportation costs. Job opportunities are poor in the smaller region precisely because it is small and its labour market is too thin for firms to hire in it.

These labour market frictions have severe consequences for workers. Together with migration costs, they prevent utility to equalise across regions. Those residing in the larger labour market are strictly better off compared to workers in the smaller labour market.

Despite that, there is no equilibrating mechanism to force regions to converge. Because of the presence of increasing returns to scale, only a large negative shock can reverse the course of divergence once it started.

From welfare perspective, the optimal outcome would be one of complete divergence where all economic activity centers in one region. However, the presence of migration costs and frictions means this will only happen when the utility difference between regions is large enough. Moreover, depending on particular parameter values, there may be large intervals on which the forces of agglomeration appear not to have any effect. In fact, the relationship between population size and the speed of divergence and migration is non-monotonous. These are the asymmetric steady states in our model where an infinitesmall shock to population does not provoke any feedback effect: the region does not shrink back to its original size, nor does it attract more migration from other regions. Instead, we observe perfect path dependence in these steady states: population rises exactly by the size of the shock. Yet in other intervals the regions may undergo adjustment, either via cross-regional hiring or migration of the unemployed.

The process of agglomeration depends in equal measure on the actions of firms and workers. The aim of this paper was to model explicitly the interactions between the two to understand how the frictions in their interaction shape regional labour markets. Understanding how strong these forces are is the area for future research.

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Appendix

8.1 Proof of proposition 1: the existence and uniqueness of equilibrium

Proof. Even though the model is set up in terms of u, v, U, in this proof we work with two derived endogenous variables, labour market activity m and labour market tightness $\theta = v/u$. We also define the average matching rate $\phi(m) = \Phi(m)/m$, and average activity per worker $\mu(\theta) = m/u = m(1,\theta)$[6]. We re-formulate the system of the three equations (5), (6), and (7) accordingly:

$$rU_i = \phi(m)\mu(\theta)\frac{\beta}{r+\delta}(y - rU_i) \tag{5*}$$

$$c\theta = \phi(m)\mu(\theta)\frac{1-\beta}{r+\delta}(y - rU_i) \tag{6*}$$

$$\delta m = \mu(\theta)[\delta P - \Phi(m)] \tag{7*}$$

We start by eliminating U from the system of the three equations (5*), (6), and (6*).

The first step is to derive the link between U and θ, by combining the first two equations:

$$\phi(m)\mu(\theta) = \frac{rU_i(r+\delta)}{\beta(y - rU_i)} = \frac{c\theta(r+\delta)}{(1-\beta)(y-rU_i)}$$

$$\therefore\ rU = \frac{\beta}{1-\beta}c\theta$$

This gives a one-to-one mapping between labour market tightness and the reservation value of unemployment. Furthermore, it gives us an upper boundary on θ: from both the job-creation curve and the reservation value curve, we can see that rU cannot exceed y, as all our endogenous variables must be non-negative. As a consequence:

$$\bar{\theta} = \frac{y}{c}\frac{1-\beta}{\beta}$$

We substitute the expression for rU from (5*) into (6*) to get a single equation in θ, m:

$$\phi(m) = \frac{\theta c(r+\delta)}{\mu(\theta)[(1-\beta)y - \beta\theta c]} \tag{A}$$

The equation pins down the locus of θ, m that is compatible with both firms' and workers' behaviour. In particular, (A) maps θ onto each m, so this equation is an implicit definition of θ as a function of market activity: $\gamma_1(m) = \theta$.

The right-hand side of equation (A) is a continuous, differentiable and increasing function of θ. Its limits are 0 for $\theta \to 0$ and ∞ for $\theta \to \bar{\theta}$. The left hand side is some function of the market activity m, bounded by 0 and ∞. As a consequence, $\gamma_1(m)$ has a solution on the $[0, \bar{\theta}]$ interval, as required.

Differentiating (A) as an implicit function of m allows us to derive the slope of γ_1:

$$\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m} = [\eta(m) - 1]\frac{\theta^2}{m}\frac{c(r+\delta)}{\phi(m)\mu(\theta)(1-\beta)y - \alpha\theta c(r+\delta)}$$

The sing of this derivative depends on the sign of $\eta(m) - 1$; in our case of globally increasing returns to scale, $\eta(m) - 1 > 0$ for all m, and γ_1' is positive. Another consequence of increasing

[6] The fact that $\Phi[m(u,v)]$ can be decomposed into m and u, v comes from the assumption of homothecity.

returns to scale at all m is that $\phi(m)$ is an increasing function of m: as market activity increases, so will the average matching rate. This means that $\gamma_1(0) = 0$, and $\gamma_1(m) \to \bar{\theta}$ as $m \to \infty$. This pins down one side of the labour market.

The other side comes from the equation describing a steady-state of unemployment inflows and outflows, equation (7*). It captures another one-to-one mapping from m to θ: $\gamma_2(m) = \theta$. For $m = 0$, $\gamma_2(0) = 0$, i.e. when there is no market activity, the number of average matches per worker is also 0, which means 0 labour market tightness. Looking at the upper boundary of the function, there must be some positive $m = \bar{m}$ for which γ_2 yields $\bar{\theta}$. To formally confirm that m, γ_2 move together, we can derive the slope of the function:

$$\frac{\mathrm{d}\gamma_2(m)}{\mathrm{d}m} = \frac{\theta\delta + \mu(\theta)\eta(m)\phi(m)\theta}{\alpha m \delta}$$

It is positive for any value of η, i.e. regardless of whether there are increasing returns to scale or not.

To summarise, the labour market equilibrium (m, θ) is determined by two functions:

- $\gamma_1(m) = \theta$, such that, for $\eta(0) > 1$, $\gamma_1(0) = 0$ and $\gamma_1(m) \to \bar{\theta}$ *as* $m \to \infty$
- $\gamma_2 = \theta$, such that $\gamma_2(0) = 0$ and $\gamma_2(\bar{m}) = \bar{\theta}$

The equilibrium lies at the intersection of these two functions. From the above, we can see there will be an equilibrium at 0 market activity: $\gamma_1(0) = \gamma_2(0) = 0$.

There will also be a second equilibrium for $m > 0$, if there is a positive m for which $\gamma_1(m) = \gamma_2(m)$. We know that $\gamma_1(\bar{m}) < \gamma_2(\bar{m}) = \bar{\theta}$; hence, for an intersection point to exist, $\gamma_1(m) > \gamma_2(m)$ for some m close to 0. Making use of our expressions for the slope of the two functions, γ_1' and γ_2', we can find the limit of their ratio as $m \to 0$. When this limit is below 1, a crossing point will exist, and there will be at least one equilibrium with positive market activity.

$$\left[\frac{m}{\gamma_1}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m} \Big/ \frac{m}{\gamma_2}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m}\right] = [\eta(m) - 1]\frac{\theta c(r+\delta)}{\phi(m)\mu(\theta)(1-\beta)y - \alpha\theta c(r+\delta)}\frac{\alpha\delta}{\delta + \mu(\theta)\eta(m)\phi(m)}$$

$$\lim_{m\to 0}\left[\frac{m}{\gamma_1}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m} \Big/ \frac{m}{\gamma_2}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m}\right] = \frac{\eta(0) - 1}{1-\alpha}\alpha = \frac{\alpha\eta(0) - \alpha}{1-\alpha}$$

The ratio of elasticities can be further simplified to give us the ratio of proportional growth. When this ratio at m close to 0 is smaller than 1, $\gamma_1 > \gamma_2$, at a crossing point exists. The analysis below shows that a sufficient condition for this to be the case is when $\alpha\eta(0) < 1$, i.e. there are decreasing returns to matching as vacancies increase.

$$\lim_{m\to 0}\left[\frac{m}{\gamma_1}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m} \Big/ \frac{m}{\gamma_2}\frac{\mathrm{d}\gamma_1(m)}{\mathrm{d}m}\right] < 1 \;\; if \;\; \alpha\eta(0) < 1 \;\; \Rightarrow \;\; \lim_{m\to 0}\left[\frac{\mathrm{d}\gamma_1(m)/\gamma_1}{\mathrm{d}\gamma_2(m)/\gamma_2}\right] < 1$$

For uniqueness of this $m > 0$ equilibrium, we need to show that the difference between the two functions, $\gamma_1(m) - \gamma_2(m)$, crosses 0 only once. We have shown that for the existence of equilibrium at $m > 0$, $\gamma_1 > \gamma_2$ close to $m = 0$, and $\gamma_1 < \gamma_2$ as $m \to \bar{m}$. Hence their difference will be falling as m rises. To ensure uniqueness, this difference must fall at all values of m: $\gamma_1' - \gamma_2' < 0$ for all m. As we demonstrate below, this will be the case when $\alpha\eta(0) < 1$, which is the same as the sufficient condition for existence of equilibrium. Hence, when $\alpha\eta(0) < 1$, there is a unique equilibrium at $m > 0$.

$$\frac{\mathrm{d}(\gamma_1 - \gamma_2)}{\mathrm{d}m} < 0 \;\Rightarrow\; \gamma_1' - \gamma_2' < 0$$

$$\begin{aligned}
\gamma_1' - \gamma_2' &\overset{\text{sgn}}{=} \frac{(\eta-1)(r+\delta)}{\phi(m)\mu(\theta)\beta + (1-\alpha)(r+\delta)} - \frac{\delta + \mu(\theta)\eta\phi(m)}{\alpha\delta} \\
&\overset{\text{sgn}}{=} \alpha\delta(\eta-1)(r+\delta) - [\phi(m)\mu(\theta)\beta + (1-\alpha)(r+\delta)][\delta + \mu(\theta)\eta\phi(m)] \\
&\overset{\text{sgn}}{=} (r+\delta)\delta[\alpha\eta(m)-1] - (1-\alpha)(r+\delta)\mu(\theta)\eta(m)\phi(m) - \phi(m)\mu(\theta)\beta[\delta + \mu(\theta)\eta(m)\phi(m)] \\
&< 0 \text{ if } \alpha\eta(m) - 1 < 0 \ \forall m
\end{aligned}$$

□

8.2 Equilibrium and population size

First, differentiate equation (). We suppress the region index i to simplify notation.

$$\begin{aligned}
rU(P) &= \frac{\Phi[m(u(P),v(P))]}{u(P)} \frac{\beta}{r+\delta}(y - rU(P)) \\
r\frac{\mathrm{d}U}{\mathrm{d}P} &= \frac{\beta}{r+\delta}(y-rU(P))\frac{1}{u^2}\left[\eta\frac{\Phi}{m}(1-\alpha)\frac{m}{u}u\frac{\mathrm{d}u}{\mathrm{d}P} + \eta\frac{\Phi}{m}\alpha\frac{m}{v}u\frac{\mathrm{d}v}{\mathrm{d}P} - \Phi\frac{\mathrm{d}u}{\mathrm{d}P}\right] - \frac{\beta}{r+\delta}\frac{\Phi}{u}r\frac{\mathrm{d}U}{\mathrm{d}P} \\
\frac{\mathrm{d}U}{\mathrm{d}P}\frac{r^2U}{y-rU} &= \frac{\mathrm{d}u}{\mathrm{d}P}\frac{rU}{u}[\eta(1-\alpha)-1] + \frac{\mathrm{d}v}{\mathrm{d}P}\frac{\eta\alpha rU}{v}
\end{aligned}$$

Next, differentiate () and collect terms.

$$\begin{aligned}
c &= \frac{\Phi[m(u(P),v(P))]}{v(P)}\frac{1-\beta}{r+\delta}(y-rU(P)) \\
0 &= \frac{1-\beta}{r+\delta}(y-rU(P))\frac{1}{v^2}\left[\eta\frac{\Phi}{m}(1-\alpha)\frac{m}{u}v\frac{\mathrm{d}u}{\mathrm{d}P} + \eta\frac{\Phi}{m}\alpha\frac{m}{v}v\frac{\mathrm{d}v}{\mathrm{d}P} - \Phi\frac{\mathrm{d}v}{\mathrm{d}P}\right] - \frac{\Phi}{v}\frac{1-\beta}{r+\delta}r\frac{\mathrm{d}U}{\mathrm{d}P} \\
r\frac{\mathrm{d}U}{\mathrm{d}P} &= \frac{\mathrm{d}u}{\mathrm{d}P}\eta(1-\alpha)\frac{y-rU}{U} + \frac{\mathrm{d}v}{\mathrm{d}P}\frac{y-rU}{v}(\eta\alpha-1)
\end{aligned}$$

Finally, we repeat the process with (7).

$$\begin{aligned}
\delta(P-u(P)) &= \Phi[m(u(P),v(P))] \\
\delta - \delta\frac{\mathrm{d}u}{\mathrm{d}P} &= \eta\frac{\Phi}{m}(1-\alpha)\frac{m}{u}\frac{\mathrm{d}u}{\mathrm{d}P} + \eta\frac{\Phi}{m}\alpha\frac{m}{v}\frac{\mathrm{d}v}{\mathrm{d}P} \\
\delta &= \frac{\mathrm{d}u}{\mathrm{d}P}\left[\eta(1-\alpha)\frac{\Phi}{u} + \delta\right] + \eta\alpha\frac{\Phi}{v}\frac{\mathrm{d}v}{\mathrm{d}P}
\end{aligned}$$

We can combine the fist two derivatives to eliminate $\frac{\mathrm{d}U}{\mathrm{d}P}$. This gives us the following relationship between $\frac{\mathrm{d}v}{\mathrm{d}P}$ and $\frac{\mathrm{d}u}{\mathrm{d}P}$:

$$\frac{\mathrm{d}u}{\mathrm{d}P} = \frac{u}{v}\frac{\mathrm{d}v}{\mathrm{d}P}$$

Combining this with the third equation, which is also in the $\frac{\mathrm{d}v}{\mathrm{d}P}, \frac{\mathrm{d}u}{\mathrm{d}P}$ space yields the solutions.

$$\begin{aligned}
\frac{\mathrm{d}u_i}{\mathrm{d}P} &= \frac{\delta u_i}{\eta\Phi^i + \delta u_i} > 0 \\
\frac{\mathrm{d}v^i}{\mathrm{d}P} &= \frac{\delta v^i}{\eta\Phi^i + \delta u_i} > 0 \\
\frac{\mathrm{d}U_i}{\mathrm{d}P} &= \frac{y - rU_i}{r}\frac{\delta}{\eta\Phi^i + \delta u_i}[\eta - 1] > 0
\end{aligned}$$